$\vec{r}(t) : \mathbb{R} \to$ vector

A **vector-valued function** is a function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where the **component functions** f, g, and h, are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\mathbf{r}(t) = \langle f(t), g(t)\rangle \quad \text{or} \quad \mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle.$$

The **limit of a vector-valued function** is equal to the limit of its component functions, provided they exist.

$$\lim_{t\to a} \langle f(t), g(t), h(t)\rangle = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

Example (Similar to Exercise 4). Evaluate

$$\lim_{t\to 3}\left[\frac{2t-4}{t+1}\mathbf{i} + \frac{t}{t^2+1}\mathbf{j} + (4t-3)\mathbf{k}\right]$$

Solution.

$$= \lim_{t\to 3}\frac{2t-4}{t+1}\mathbf{i} + \lim_{t\to 3}\frac{t}{t^2+1}\mathbf{j} + \lim_{t\to 3}(4t-3)\mathbf{k} = \boxed{\frac{1}{2}\mathbf{i} + \frac{3}{10}\mathbf{j} + 9\mathbf{k}}$$

Example (Similar to Exercise 6). Given the vector-valued function $\mathbf{r}(t) = \langle t^2 - 4, 4t+3\rangle$, find the following values:

(a) $\lim_{t\to 3}\mathbf{r}(t)$

Answer. $\langle 5, 15\rangle$

(b) $\mathbf{r}(3)$

Answer. $\langle 5, 15\rangle$

(c) Is $\mathbf{r}(t)$ continuous at $x = 3$?

Answer. Yes

(d) $\mathbf{r}(t+1) - \mathbf{r}(t)$

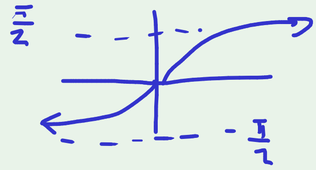
Solution.

$$\begin{aligned} &= \left\langle (t+1)^2 - 4, 4(t+1) + 3\right\rangle - \left\langle t^2 - 4, 4t+3\right\rangle \\ &= \left\langle t^2 + 2t + 1 - 4 - (t^2-4), 4t + 4 + 3 - (4t+3)\right\rangle \\ &= \boxed{\langle 2t+1, 4\rangle} \end{aligned}$$

Example (Similar to Exercise 12)**.** Find the limit of the vector-valued function at the indicated value of t:

$$\lim_{t\to\infty}\left\langle e^{-2t}, \frac{2t+3}{3t-1}, \arctan(2t)\right\rangle$$

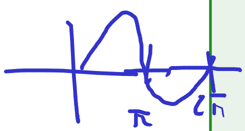
$\lim_{t\to\infty} \frac{1}{e^{2t}} = \frac{1}{\infty} = 0$



Answer.

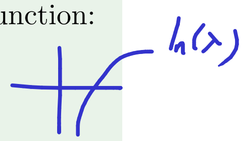
$$\boxed{\left\langle 0, \frac{2}{3}, \frac{\pi}{2}\right\rangle}$$

Example (Similar to Exercise 16)**.** Find the domain of the vector-valued function:



$\frac{1}{\sin t}$

$$\mathbf{r}(t) = \left\langle \csc(t), \frac{1}{\sqrt{t-3}}, \ln(t-2)\right\rangle$$



$\sin t = 0 \Rightarrow k\pi$ $t > 3$ $t > 2$

Answer. $t > 3$ and $t \neq n\pi$, where $n \in \mathbb{Z}$

Example (Similar to Exercise 23)**.** Eliminate the parameter t, write the equation in Cartesian coordinates, then sketch the graph of the vector-valued function:

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j}$$

$x(t)$, $y(t)$

Solution.

$$x = 2t \implies \frac{x}{2} = t$$

$$y = t^2 = \frac{x^2}{4}$$

$$\boxed{y = \frac{x^2}{4}}$$

Plot: