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1 Derivatives of Vector-Valued Functions

Definition. The **derivative of a vector-valued function** $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

provided the limit exists.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

Example (Similar to Exercise 42)**.** Compute the derivative of the vector-valued function

$$\mathbf{r}(t) = e^{-t}\mathbf{i} + \sin(3t)\mathbf{j} + 10\sqrt{t}\mathbf{k}$$

$$\sqrt{x} \xrightarrow{d} \frac{1}{2\sqrt{x}}$$

$$x^{\frac{1}{2}} \xrightarrow{d} \frac{1}{2} x^{-\frac{1}{2}}$$

Answer.

$$\mathbf{r}'(t) = -e^{-t}\mathbf{i} + 3\cos(3t)\mathbf{j} + \frac{5}{\sqrt{t}}\mathbf{k}$$

1.1 Properties of the Derivative of Vector-Valued functions

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

Linearity

$$\frac{\mathrm{d}}{\mathrm{d}t}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

Product rules

$(fg)' = f'g + fg'$

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t)\cdot\mathbf{u}(t)] = \mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(t)\times\mathbf{u}(t)] = \mathbf{r}'(t)\times\mathbf{u}(t) + \mathbf{r}(t)\times\mathbf{u}'(t)$$

← order matters

Chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))\cdot f'(t)$$

$f(g(t)) \xrightarrow{d} f'(g(t))\cdot g'(t)$

2 Tangent Vectors and Unit Tangent Vectors

Definition. Let C be a curve defined by a vector-valued function $\mathbf{r}$. A **tangent vector** $\mathbf{v}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, the vector $\mathbf{v}$ is tangent to the curve C. The **principal unit tangent vector** at t is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

provided $\|\mathbf{r}'(t)\| \neq 0$.

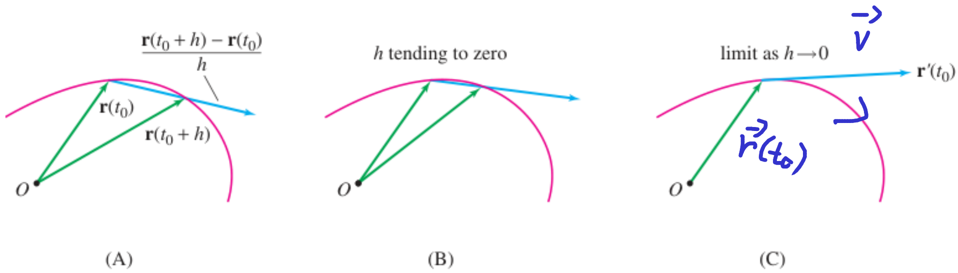

FIGURE 3 The difference quotient converges to a vector $\mathbf{r}'(t_0)$, tangent to the curve.

Example (Similar to Exercise 56)**.** Find the unit tangent vector for the following parametrized curve:

$$\mathbf{r}(t) = 3\cos(4t)\mathbf{i} + 3\sin(4t)\mathbf{j} + 5t\mathbf{k}, \quad 1 \le t \le 2$$

Solution.

$\sin^2 + \cos^2 = 1$

$$\begin{aligned}
\mathbf{r}'(t) &= -12\sin(4t)\mathbf{i} + 12\cos(4t)\mathbf{j} + 5\mathbf{k} \\
\|\mathbf{r}'(t)\| &= \sqrt{144\sin^2(4t) + 144\cos^2(4t) + 25} \\
&= \sqrt{144 + 25} = 13 \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \boxed{-\frac{12}{13}\sin(4t)\mathbf{i} + \frac{12}{13}\cos(4t)\mathbf{j} + \frac{5}{13}\mathbf{k}}
\end{aligned}$$

3 Integrals of Vector-Valued Functions

Definition.

$$\int \langle f(t), g(t), h(t) \rangle \, \mathrm{d}t = \left\langle \int f(t)\, \mathrm{d}t, \int g(t)\, \mathrm{d}t, \int h(t)\, \mathrm{d}t \right\rangle$$

$$\int_a^b \langle f(t), g(t), h(t) \rangle \, \mathrm{d}t = \left\langle \int_a^b f(t)\, \mathrm{d}t, \int_a^b g(t)\, \mathrm{d}t, \int_a^b h(t)\, \mathrm{d}t \right\rangle$$

Example (Similar to Exercise 64)**.** The acceleration function, initial velocity, and initial position of a particle are

$$\begin{aligned} \mathbf{a}(t) &= 2\sin t\mathbf{i} - 3\cos t\mathbf{j} \\ \mathbf{v}(0) &= 3\mathbf{i} + 4\mathbf{j} \\ \mathbf{r}(0) &= 5\mathbf{i} + 6\mathbf{j} \end{aligned}$$

Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

Solution. Since $\mathbf{v}'(t) = \mathbf{a}(t)$,

$$\begin{aligned} \mathbf{v}(t) &= \int \mathbf{a}(t)\,\mathrm{d}t = -2\cos t\mathbf{i} - 3\sin t\mathbf{j} + \mathbf{c} \\ \mathbf{v}(0) &= 3\mathbf{i} + 4\mathbf{j} = -2\mathbf{i} + \mathbf{c} \\ &\implies \mathbf{c} = 5\mathbf{i} + 4\mathbf{j} \\ \mathbf{v}(t) &= \boxed{(-2\cos t + 5)\mathbf{i} + (-3\sin t + 4)\mathbf{j}} \end{aligned}$$

Since $\mathbf{r}'(t) = \mathbf{v}(t)$,

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t)\,\mathrm{d}t = (-2\sin t + 5t)\mathbf{i} + (3\cos t + 4t)\mathbf{j} + \mathbf{c} \\ \mathbf{r}(0) &= 5\mathbf{i} + 6\mathbf{j} = 0\mathbf{i} + 3\mathbf{j} + \mathbf{c} \\ &\implies \mathbf{c} = 5\mathbf{i} + 3\mathbf{j} \\ \mathbf{r}(t) &= \boxed{(-2\sin t + 5t + 5)\mathbf{i} + (3\cos t + 4t + 3)\mathbf{j}} \end{aligned}$$