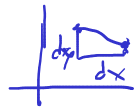
Contents

1 Arc Length for Vector Functions

ds



Plane curve For a smooth curve C defined by the function $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, where $t \in [a, b]$, the arclength of C over the interval is

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, \mathrm{d}t = \int_a^b \left\|\mathbf{r}'(t)\right\| \mathrm{d}t$$

Space curve For a smooth curve C defined by the function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where $t \in [a, b]$, the arclength of C over the interval is

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, \mathrm{d}t = \int_a^b \left\|\mathbf{r}'(t)\right\| \mathrm{d}t$$

Example (Similar to Exercise 102)**.** Find the arc length of the curve on the given interval.

$$\mathbf{r}(t) = \langle t\cos t, t\sin t, 2t\rangle\,, \qquad 0 \le t \le 2\pi$$

Solution.

$$\begin{aligned}
\mathbf{r}'(t) &= \langle \cos t - t\sin t, \sin t + t\cos t, 2\rangle \\
\left\|\mathbf{r}'(t)\right\| &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 4} \\
&= \sqrt{(\cos^2 t - \cancel{2t\sin t\cos t} + t^2\sin^2 t) + (\sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t) + 4} \\
&= \sqrt{t^2 + 5}
\end{aligned}$$

so the arclength is computed by the integral $\displaystyle\int_0^{2\pi} \sqrt{t^2+5}\,\mathrm{d}t$.
Performing this integral from scratch is a bit tedious. In general, one has the formula

$$\int \sqrt{u^2 + a^2}\,\mathrm{d}u = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2+a^2}\right| + C$$

(for a derivation of this, see my writeup: https://www.overleaf.com/read/nzhzfwfwrxkc) so we have

$$\begin{aligned}\int_0^{2\pi} \sqrt{t^2+5}\,\mathrm{d}t &= \left[\frac{t}{2}\sqrt{t^2+5} + \frac{5}{2}\ln\left|t+\sqrt{t^2+5}\right|\right]_0^{2\pi} \\ &= \boxed{\pi\sqrt{4\pi^2+5} + \frac{5}{2}\ln\left(2\pi+\sqrt{4\pi^2+5}\right) - \frac{5}{2}\ln\sqrt{5}} \\ &\approx 25.343.\end{aligned}$$

2 Arclength Parametrization

Let $\mathbf{r}(t)$ describe a smooth curve for $t \geq a$. Then the arclength function is given by

$$s(t) = \int_a^t \left\|\mathbf{r}'(u)\right\| \mathrm{d}u$$

This gives us a function from t to s. The inverse function will be a function from s to t, which we can compose with $\mathbf{r}(t)$ to construct the arclength parametrization $\mathbf{r}(s)$.

Example. Find the arclength parametrization for the curve

$$\mathbf{r}(t) = \langle t+3, 2t-4, 2t\rangle, \quad t \geq 3$$

Solution.

$$\begin{aligned}\mathbf{r}'(t) &= \langle 1,2,2\rangle \\ \left\|\mathbf{r}'(t)\right\| &= \left\|\langle 1,2,2\rangle\right\| = \sqrt{1+4+4} = 3 \\ s(t) &= \int_a^t \left\|\mathbf{r}'(u)\right\| \mathrm{d}u = \int_3^t 3\,\mathrm{d}u = 3u\Big|_{u=3}^{t} = 3t-9\end{aligned}$$

so $s = 3t-9 \implies t = \frac{s}{3}+3$. Plugging into the original parametrization gives the arclength parametrization:

$$\begin{aligned}\mathbf{r}(s) &= \left\langle \left(\frac{s}{3}+3\right)+3,\ 2\left(\frac{s}{3}+3\right)-4,\ 2\left(\frac{s}{3}+3\right)\right\rangle \\ &= \boxed{\left\langle \frac{s}{3}+6,\ \frac{2s}{3}+2,\ \frac{2s}{3}+6\right\rangle}\end{aligned}$$

3 Curvature

The notion of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the curve, the greater the curvature.

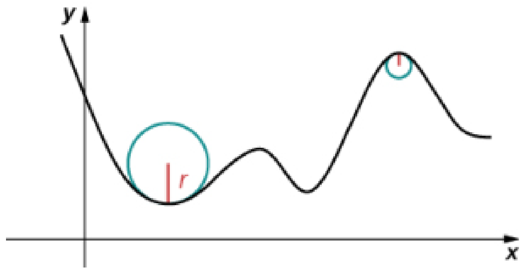


Figure 3.6 The graph represents the curvature of a function $y = f(x)$. The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Definition. Let C be a smooth curve in the plane or in space given by $\mathbf{r}(s)$, where s is the arc-length parameter. The **curvature** κ at s is

$$\kappa = \left\| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right\| = \left\| \mathbf{T}'(s) \right\|.$$

This formula is not very easy to use. Note that previously we defined $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|}$. To use the formula for curvature, we would have to

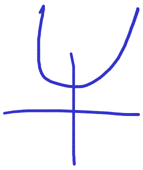
1. express $\mathbf{r}(t)$ in terms of the arc-length parameter s, to get $\mathbf{r}(s)$
2. compute the unit tangent vector $\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\left\|\mathbf{r}'(s)\right\|}$
3. take the derivative of $\mathbf{T}(s)$ with respect to s, to get $\mathbf{T}'(s)$
4. take its magnitude to get $\left\|\mathbf{T}'(s)\right\| = \kappa$

This is a tedious process.

Alternative Formulas for Curvature

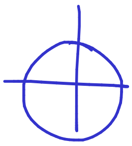
- If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|}$$



- If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\left\|\mathbf{r}'(t) \times \mathbf{r}''(t)\right\|}{\left\|\mathbf{r}'(t)\right\|^3}$$



- If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$$

For proofs of these formulas, see the OpenStax Calc 3 textbook, under Theorem 3.6.

Example. Show that the curvature of a circle of radius r is $\kappa = \frac{1}{r}$.

Solution. One can parametrize a circle of radius r as

$$\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle\,, \quad t \in [0, 2\pi],$$

and compute

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -r\sin t, r\cos t\rangle \\
\left\|\mathbf{r}'(t)\right\| &= \sqrt{r^2\sin^2 t + r^2\cos^2 t} = \sqrt{r^2} = r \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|} = \frac{\langle -r\sin t, r\cos t\rangle}{r} = \langle -\sin t, \cos t\rangle \\
\mathbf{T}'(t) &= \langle -\cos t, -\sin t\rangle \\
\kappa &= \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|} = \boxed{\frac{1}{r}}
\end{aligned}$$

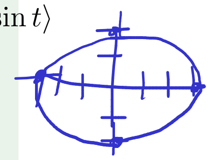
Example (Similar to Exercise 126)**.** Find the unit tangent vector $\mathbf{T}(t)$ for $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 3t\rangle$.

Solution.

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|} \\
&= \frac{\langle -4\sin t, 4\cos t, 3\rangle}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\
&= \left\langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\right\rangle.
\end{aligned}$$

$\sqrt{16+9} = \sqrt{25} = 5$

Example (Similar to Exercise 130)**.** Find the curvature of the curve $\mathbf{r}(t) = \langle 3\cos t, 2\sin t\rangle$ at $t = \frac{\pi}{6}$. (Note that this is an ellipse)



Solution. We use the formula that $\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$. In order to take the cross product, we treat the xy-plane that this curve lives on as having the z coordinate $= 0$.

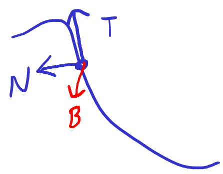
$$
\begin{aligned}
\mathbf{r}(t) &= \langle 3\cos t, 2\sin t, 0\rangle \\
\mathbf{r}'(t) &= \langle -3\sin t, 2\cos t, 0\rangle \\
\mathbf{r}''(t) &= \langle -3\cos t, -2\sin t, 0\rangle \\
\mathbf{r}' \times \mathbf{r}'' &= \left\langle 0, 0, 6\sin^2 t + 6\cos^2 t\right\rangle = \langle 0, 0, 6\rangle \\
\kappa &= \left.\frac{6}{(9\sin^2 t + 4\cos^2 t)^{3/2}}\right|_{t=\frac{\pi}{6}} = \boxed{\frac{16}{7\sqrt{21}}}
\end{aligned}
$$

4 Normal and Binormal

Let C be a three-dimensional smooth curve represented by $\mathbf{r}$ over an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** at t is defined to be

T(t)

$$
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}
$$



The **binormal vector** at t is defined as

$$
\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t),
$$

where $\mathbf{T}(t)$ is the unit tangent vector.

Remark. The binormal vector is orthogonal to both the unit tangent vector and the normal vector. It is also always a unit vector.