Contents

1 Functions of two variables

Definition. A **function of two variables** $f(x, y) = z$ maps a point $(x, y) \in \mathbb{R}^2$ to a real number z. Like with functions of one variable, the **domain** of a function is the possible inputs for the function, and the **range** is the possible outputs for the function.

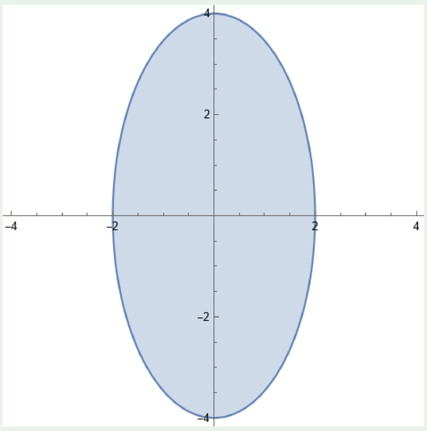
Example (Similar to Exercises 6, 13)**.** Find the domain and range of the function

$$g(x, y) = \sqrt{16 - 4x^2 - y^2}.$$

Solution. The range is easier to see. The minimum obtainable value is 0, the maximum is ~~16~~ 4 (when $x, y = 0$). So the range is $[0, 4]$. The domain comes from the domain of the square-root function. We must have $16 - 4x^2 - y^2 \geq 0 \implies 4x^2 + y^2 \leq 16$. This final inequality defines an ellipse depicted below, which is the domain of the function.

Domain$(g) =$ $\{x, y \in \mathbb{R}^2 \mid 4x^2 + y^2 \leq 16\}$

$$\frac{x^2}{4} + \frac{y^2}{16} \leq 1$$



1.1 Graphing functions of two variables

The graph of a function of two variables $f(x, y)$ is a **surface** (a two-dimensional object sitting in three-dimensional space).

Example.

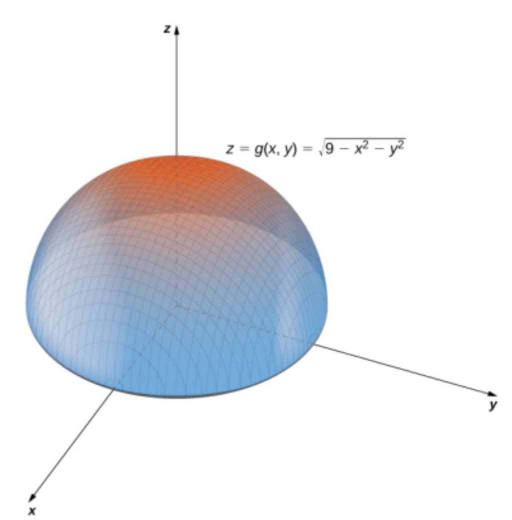


Figure 4.4 Graph of the hemisphere represented by the given function of two variables.

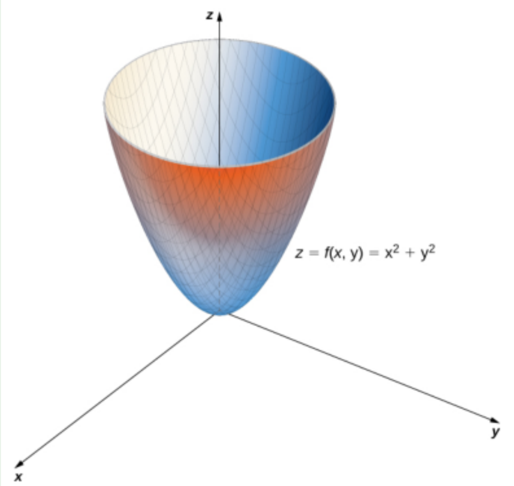


Figure 4.5 A paraboloid is the graph of the given function of two variables.

1.2 Level curves

Definition. Given a function $f(x, y)$ and a number c in the range of f, a **level curve of a function of two variables** for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

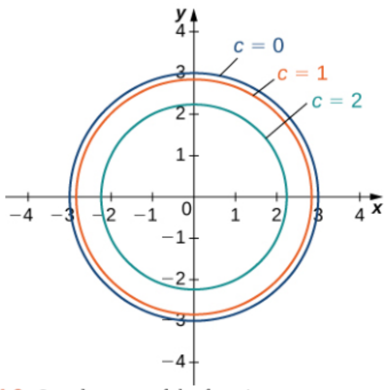


Figure 4.8 Level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, using $c = 0, 1, 2,$ and 3 ($c = 3$ corresponds to the origin).

A graph of the various level curves of a function is called a **contour map**.

Another useful tool for understanding the graph of a function of two variables is called a vertical trace. Level curves are alwasy graphed in the xy-plane. Vertical traces are graphed in the xz- or yz-planes.

Definition. A **vertical trace** of a function can be either

- the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$, or
- the set of points that solves the equation $f(x, b) = z$ for a given constant $y = b$

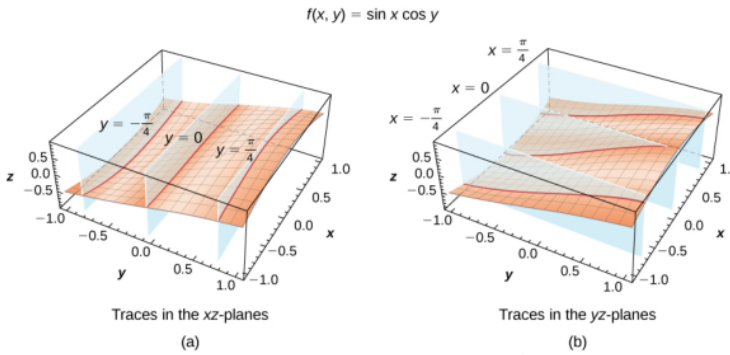


Figure 4.11 Vertical traces of the function $f(x, y)$ are cosine curves in the xz-planes (a) and sine curves in the yz-planes (b).

Example (Similar to Exercise 18, 20)**.** Find the level curves of the function at the indicated value of c to visualize the given function.

$$f(x, y) = xy - x \quad \text{for } c = -2, -1, 0, 1, 2 :$$

$x(y-1) = 0$

$\Rightarrow x = 0$ or $y = 1$

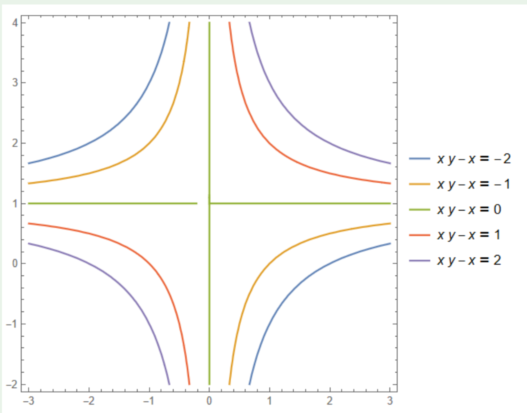
$x(y-1) = 2$

$y = \frac{2}{x} + 1$

HA: $y = 1$

VA: $x = 0$

Answer.



2 Functions of more than two variables

In particular, a function of 3 variables will be relevant to us in calc 3. Such functions take three inputs and return one output.

To visualize the graph of such a function would require four dimensions.

Instead, like with functions of two variables, we can take cross sections (level curves / traces).

Functions of two variables have level curves. Functions of three variables have level *surfaces*.

Definition. Given a function $f(x, y, z)$ and a number c in the range of f, a **level surface of a function of three variables** is defined to be the set of points satisfying the equation $f(x, y, z) = c$.

Example (Similar to Exercise 50)**.** Find the level surfaces for the function of three variables and describe them: $f(x, y, z) = 4x^2 + 9y^2 - z^2$.

$$\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{9}} - \frac{z^2}{1} = 1$$

Solution.

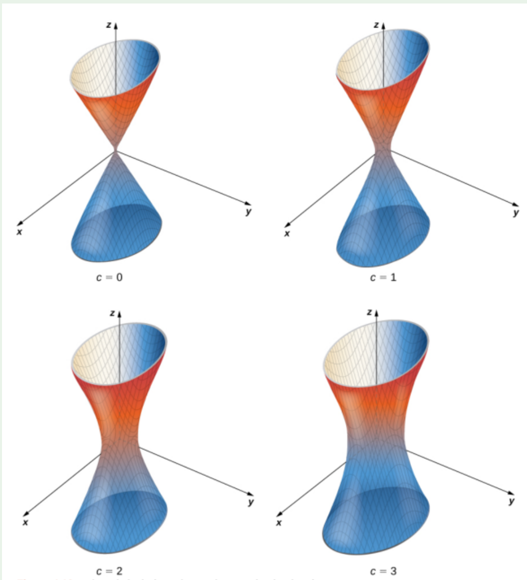


Figure 4.13 A hyperboloid of one sheet with some of its level surfaces.

The level surfaces are:

- $c = 0$: elliptic cone
- $c > 0$: one-sheet hyperboloid
- $c < 0$: two-sheet hyperboloid