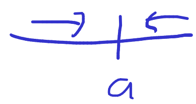
Contents

1 Limit in Calc 1 versus Limit in Calc 3

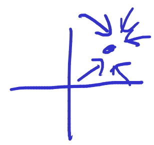
In Calc 1, to define the limit at a point, $\lim_{x\to a} f(x)$, we introduced the left-hand limit and right hand limit, and said that the (two-sided) limit exists if the left-hand limit equals the right-hand limit and that it takes that common value:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) =: \lim_{x\to a} f(x)$$



Note that we only had two direction to approach $x = a$ along the number line.

In Calc 3, with functions of two (or potentialy more) variables, there are infinitely many more ways to approach a point in the plane (a, b) or in space (a, b, c). The definition of limit has to account for *all* possible directions to approach a given point. (It turns out that even considering all linear paths approaching the point is not enough. All possible directions means all possible *curves*.)



2 Limit of a function of two variables

The idea we want to capture is simple, but the formalism is a bit abstract. The idea is:

> $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if no matter what direction we approach (a, b), the function tends toward the value L.

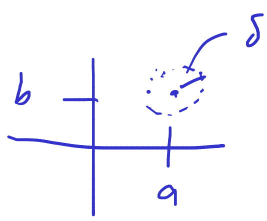
The difficulty lies in expressing "in every direction" mathematically.

To do this, we have a formalism involving δ- and ε- disks/neighborhoods.

Feel free to skip to the section on continuity. You don't need to know this.

Definition. Consider a point $(a, b) \in \mathbb{R}^2$. A **δ-disk** centered at a point (a, b) is defined to be an open disk of radius δ centered at point (a, b)—that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < \delta^2\}$$



Definition. Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x,y) in a δ disk around (a, b), except possibly for (a,b) itself, the value of $f(x,y)$ is no more than ε away from L (Figure 4.15). Using symbols, we write the following:
For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

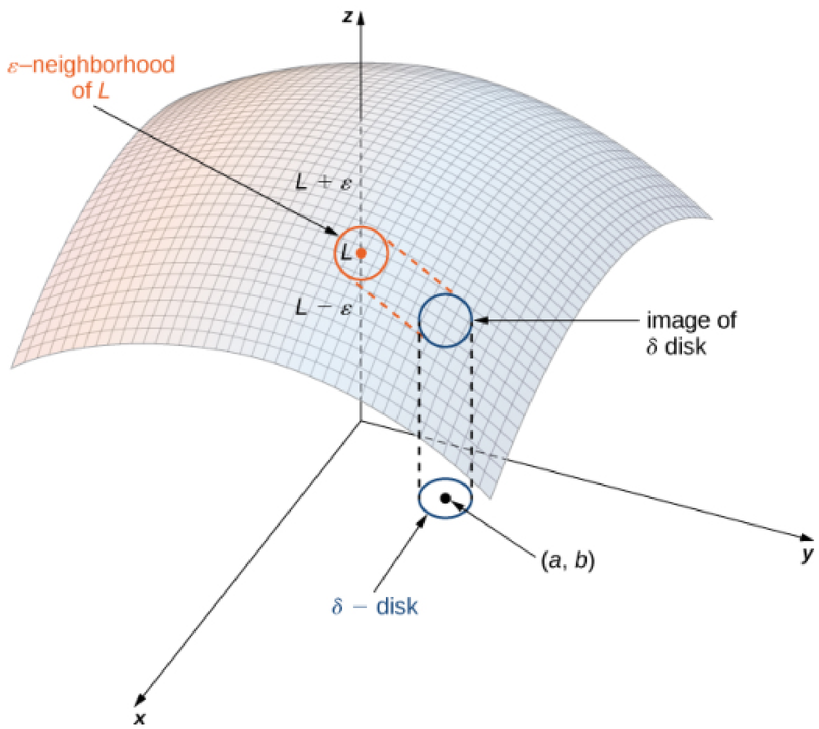


Figure 4.15 The limit of a function involving two variables requires that $f(x, y)$ be within ε of L whenever (x, y) is within δ of (a, b). The smaller the value of ε, the smaller the value of δ.

Thankfully, you will not need to work with such definitions (unless you are a math major and eventually take a course on "Real Analysis")

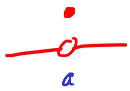
3 Continuity of functions of two variables

Recall from Calc 1, that a function of one variable $f(x)$ is continuous at a point $x = a$ if:

(i) $f(a)$ exists.

(ii) $\lim_{x \to a} f(x)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$.



These three conditions similarly define continuity of a function of two variables.

Definition. A function $f(x, y)$ is **continuous at a point** (a, b) in its domain if the following conditions are satisfied:

(i) $f(a, b)$ exists.

(ii) $\lim_{(x,y) \to (a,b)} f(x, y)$ exists.

(iii) $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$.

Nice properties

- The sum of continuous functions is continuous
- The product of continuous functions is continuous
- The composition of continuous functions is continuous

4 Limit Laws

All the expected limit laws carry over from Calc 1:

If $\lim_{(x,y) \to (a,b)} f(x) = L$ and $\lim_{(x,y) \to (a,b)} g(x) = M$, and c is a constant, then limits satisfy the following:

- Constant Law:

$$\lim_{(x,y) \to (a,b)} c = c$$

- Identity Laws:

$$\lim_{(x,y) \to (a,b)} x = a, \qquad \lim_{(x,y) \to (a,b)} y = b$$

- Sum / Difference Law:

$$\lim_{(x,y) \to (a,b)} (f(x, y) \pm g(x, y)) = L \pm M$$

- Constant Multiple Law:
$$\lim_{(x,y)\to(a,b)} (cf(x,y)) = cL$$
- Product Law:
$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM$$
- Quotient Law:
$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$$
for $M \neq 0$
- Power Law:
$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$$
for any positive integer n
- Root Law:
$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$
for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

5 Examples of good behavior

Example (Similar to Exercise 61, 68)**.** Compute the limit
$$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y}$$

Solution. We can just plug in the point since the function is continuous at this point.
$$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y} = \frac{5}{6}$$

Example (Similar to Exercise 82)**.** Determine if the following limit exists or not. If it does exist give the value of the limit.
$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2}$$

Solution. Notice that we can factor the numerator and denominator as follows

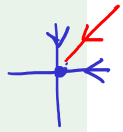
$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{(2x+y)(x-y)}{(x+y)(x-y)} = \lim_{(x,y)\to(1,1)} \frac{(2x+y)}{(x+y)}$$

The cancellation results in a function that we can take the limit of. Taking the limit gives

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{(2x+y)}{(x+y)} = \frac{3}{2}$$

6 Examples of bad behavior

Example. Show that the limit



$$\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2 + y^2}$$

does not exist

Solution. First consider appproaching along the line $y = 0$. Approaching along this line, the limit simplifies to

$$\lim_{x\to 0} \frac{0}{3x^2} = 0$$

Note that the same happens if we approach along the line $x = 0$. However, if we approach along the line $y = x$, the limit simplifies to

$$\lim_{x\to 0} \frac{2x^2}{4x^2} = \frac{1}{2}$$

We exhibited two different ways to approach the point $(0, 0)$ which resulted in different values, hence the limit cannot exist.

Example (Along all lines is not enough; Similar to Exercise 88)**.** Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4}$$

does not exist.

Solution. Trying to approach along the x- and y-axes, the limit simplifies to yield 0. If we approach along a line $y = kx$, the limit becomes

$$\lim_{x\to 0} \frac{4x(kx)^2}{x^2 + 3(kx)^4} = \lim_{x\to 0} \frac{4k^2x^3}{x^2 + 3k^4x^4} = \lim_{x\to 0} \frac{4k^2x}{1 + 3k^4x^2} = \frac{0}{1+0} = 0$$

But if we approach along the curve $x = y^2$, the limit simplifies to

$$\lim_{y \to 0} \frac{4y^2y^2}{(y^2)^2 + 3y^4} = \lim_{y \to 0} \frac{4y^4}{4y^4} = 1$$

We exhibited two different ways to approach the point $(0, 0)$ which resulted in different values, hence the limit cannot exist.