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1 What is a partial derivative?

Like in Calc 1, a derivative measures the rate of change of a function. But in Calc 1, we dealt with functions of one variable, there was only one possible direction of change (positive x-direction).

In Calc 3, we extend this notion to functions of 2 and 3 variables (and potentially more). For such functions, we have more ways (directions) to vary the input of our function. Thus our notion of derivative will need to to emphasize what direction we vary the input. The result is the notion of a *partial* derivative, where we measure the rate of change of the function as we vary one of the input variables and leave the others as fixed. (It will also lead us to the notion of a *directional* derivative, but we'll wait till §4.6 to define that)

clean up this..

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1.1 In dimension 2

Definition. Let $f(x, y)$ be a function of two variables.

- The **partial derivative of f with respect to x**, written as $\frac{\partial f}{\partial x}$ or f_x, is defined as
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$
- The **partial derivative of f with respect to y**, written as $\frac{\partial f}{\partial y}$ or f_y, is defined as
$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

1.2 In dimension 3

Definition. Let $f(x, y, z)$ be a function of three variables.

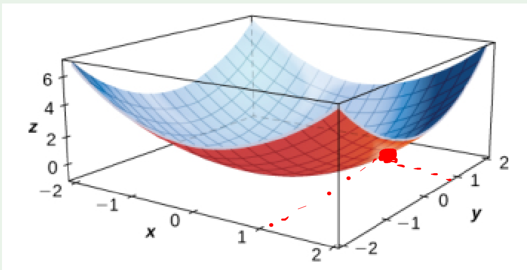
- The **partial derivative of f with respect to x**, written as $\frac{\partial f}{\partial x}$, or f_x, is defined to be
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

- The **partial derivative of f with respect to y**, written as $\frac{\partial f}{\partial y}$, or f_y, is defined to be
$$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k}$$

- The **partial derivative of f with respect to z**, written as $\frac{\partial f}{\partial z}$, or f_z, is defined to be
$$\frac{\partial f}{\partial z} = \lim_{m\to 0} \frac{f(x, y, z+m) - f(x, y, z)}{m}.$$

The main idea: When taking a partial derivative, treat the other variables as constant. Then use the derivative rules you have from Calc 1.

Example (Similar to Exercise 115). Calculate the sign of the partial derivative using the graph of the surface:



$f_x(1,1)$ and $f_y(1,1)$.

Answer. Both are positive.

Example (Similar to Exercise 120). Calculate the partial derivative $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = \ln\left(x^6 + y^4\right)$

Answer.

$$\frac{\partial z}{\partial x} = \frac{6x^5}{x^6+y^4}$$
$$\frac{\partial z}{\partial y} = \frac{4y^3}{x^6+y^4}$$

Example (Similar to Exercise 124)**.** Let $z = \tan(2x - y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer.

$$\frac{\partial z}{\partial x} = \sec^2(2x - y) \cdot 2$$
$$\frac{\partial z}{\partial y} = \sec^2(2x - y) \cdot -1$$

2 Higher-Order Partial Derivatives

Like in Calc 1, where we could take the derivative of the derivative of a function (the second derivative), and higher (f'', f''', etc.), the same can be done with partial derivatives.

With a function of two variables, $f(x, y)$, there are four partial derivatives:

$$\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} \\
f_{xy} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \\
f_{yx} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \\
f_{yy} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

The higher order partial derivatives with different variables (e.g., f_{xy}, f_{yx}) are often called **mixed partial derivatives**.

Note that the order of the letters switch from the first column to the third. The order ends up not mattering (assuming continuity of the function) thanks to Clairaut's theorem:

Theorem (Clairaut's Theorem: Equality of Mixed Partial Derivatives)**.** Suppose that $f(x, y)$ is defined on an open disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$

Remark. For the purposes of this course, we can assume that Clairaut's Theorem always holds. (Counterexamples exist, but they shouldn't show up.)

Example (Similar to Exercise 138). Verify Clairaut's theorem for the function $f(x, y, z) = e^{-2x} \sin(z^2 y)$

Solution. Computing first order partials:

$$\frac{\partial f}{\partial x} = -2e^{-2x} \sin(z^2 y)$$
$$\frac{\partial f}{\partial y} = e^{-2x} \cos(z^2 y) \cdot z^2$$

Computing the mixed partials:

$$\frac{\partial^2 f}{\partial x \partial y} = -2e^{-2x} \cos(z^2 y) z^2$$
$$\frac{\partial^2 f}{\partial y \partial x} = -2e^{-2x} \cos(z^2 y) z^2$$

Example. Find f_{xxyzz} for $f(x, y, z) = z^3 y^2 \ln x$

Solution.

$$f_x = \frac{z^3 y^2}{x}$$
$$f_{xx} = -\frac{z^3 y^2}{x^2}$$
$$f_{xxy} = -\frac{2z^3 y}{x^2}$$
$$f_{xxyz} = -\frac{6z^2 y}{x^2}$$
$$f_{xxyzz} = -\frac{12zy}{x^2}$$