Contents

1 Recall the related Calc 1 notions

Recall in Calc 1, for functions of one variable, you learned the notion of a **tangent line** at a point $x = a$ and **linear approximation** at a point $x = a$. You learned that these were different ways of viewing the same concept, with the formula

$(a, f(a))$

$$L(x) = f(a) + f'(a)(x - a).$$

This can be generalized to functions of two variables.

2 Tangent Plane / Linear Approximation

Definition. Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Definition. Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the **linear approximation** of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example (Similar to Exercise 171)**.** Find the equation for the tangent plane to $z = \ln(2x + y)$ at $(-1, 3)$.

Solution.

$$\begin{aligned} f(x,y) &= \ln(2x+y) & f(-1,3) &= \ln(1) = 0 \\ f_x(x,y) &= \frac{2}{2x+y} & f_x(-1,3) &= \frac{2}{1} = 2 \\ f_y(x,y) &= \frac{1}{2x+y} & f_y(-1,3) &= \frac{1}{1} = 1 \end{aligned}$$

The equation of the plane is

$$\boxed{z = 0 + 2(x+1) + 1(y-3)}$$

Example (Similar to Exercise 174)**.** Find the equation for the tangent plane to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4, 3)$.

Solution.

$$\begin{aligned} f(x,y) &= 3 + \frac{x^2}{16} + \frac{y^2}{9} & f(-4,3) &= 3 + 1 + 1 = 5 \\ f_x(x,y) &= \frac{x}{8} & f_x(-4,3) &= -\frac{1}{2} \\ f_y(x,y) &= \frac{2y}{9} & f_y(-4,3) &= \frac{2}{3} \end{aligned}$$

The equation of the plane is

$$\boxed{z = 5 - \frac{1}{2}(x+4) + \frac{2}{3}(y-3)}$$

3 Differentials

Recall from Calc 1 that we had the notion of differentials. The differential of y was defined as $\mathrm{d}y = f'(x)\,\mathrm{d}x$. The differential was used to approximate $\Delta y = f(x + \Delta x) - f(x)$, where $\Delta x = \mathrm{d}x$.

Extending this idea to the linear approximation of a function of two variables at a point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Definition. Letting $\mathrm{d}x = \Delta x$ and $\mathrm{d}y = \Delta y$, then the differential $\mathrm{d}z$, also called the **total differential** of $z = f(x, y)$ at (x_0, y_0) is defined as

$$\mathrm{d}z = f_x(x_0, y_0)\,\mathrm{d}x + f_y(x_0, y_0)\,\mathrm{d}y.$$

Example (Similar to Exercise 192)**.** Find the total differential of the function

$$u = \frac{t^3 r^6}{s^2}$$

Answer.

$$\mathrm{d}u = \frac{3t^2 r^6}{s^2}\,\mathrm{d}t + \frac{6t^3 r^5}{s^2}\,\mathrm{d}r - \frac{2t^3 r^6}{s^3}\,\mathrm{d}s$$