Contents

1 Revisiting the Calc 1 chain rule

In Calc 1, we learned the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x)$$

To be able to relate this to the Calc 3 version of chain rule, we want to think of f as a function depending on g, and g as a function depending on x. Written mathematically,

$$f(g) \quad \text{and} \quad g(x)$$

They have a dependancy relation / tree diagram

$$f \longrightarrow g \longrightarrow x$$

Reading off the chain rule from the diagram,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

This looks strange, but it identical to the chain rule we're used to. The factor $f'(g(x))$ can be thought of as $\frac{\mathrm{d}f}{\mathrm{d}x} \circ g$, which means we take the derivative of f with respect to x, and then replace x with g. This replacement is a variable rewrite, so we end up rewriting $\frac{\mathrm{d}f}{\mathrm{d}x}$ as $\frac{\mathrm{d}f}{\mathrm{d}g}$.

There is a perspective shift. The Calc 1 version treats the composition $f \circ g$ as a separate function, built up from the derivatives of the constituent functions f and g and asks "how does this composition $f \circ g$ vary as x varies":

$$\frac{\mathrm{d}}{\mathrm{d}x}(f \circ g) = \left(\frac{\mathrm{d}f}{\mathrm{d}x} \circ g\right) \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

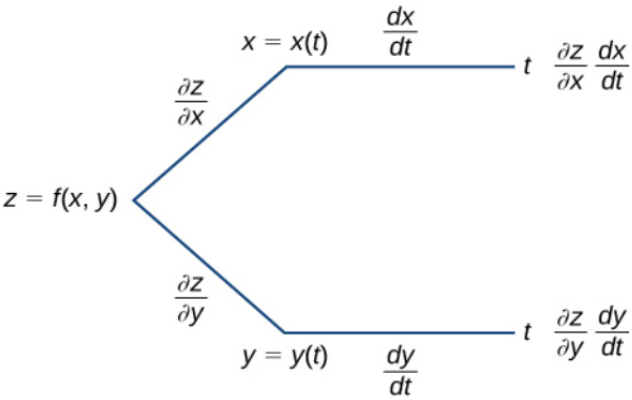
The Calc 3 version treats the nesting of the function g with f as being native to f and instead asks "how does f vary as x varies":

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

2 Generalized chain rule

We build up to the general chain rule with some smaller examples.

- Suppose $f(x, y)$, where $x = g(t)$ x(t) and $y = h(t)$ y(t). Then $z = f(x(t), y(t))$ is a function of t. The tree diagram is



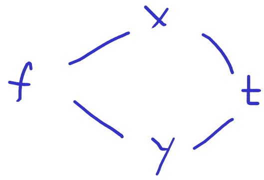
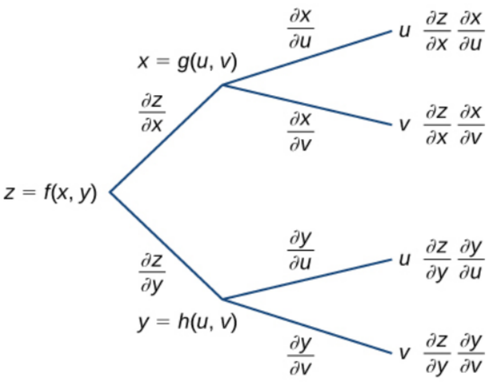


Figure 4.34 Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

from which we get the chain rule:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

- Suppose $f(x, y)$, where $x = g(u, v)$ and $y = h(u, v)$. Then, $z = f(g(u, v), h(u, v))$ is a function of u and v. Considering the tree diagram:



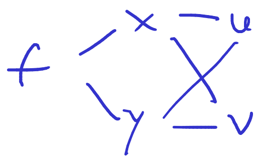


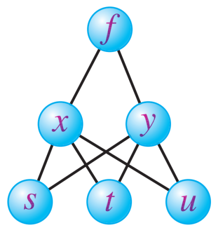
Figure 4.35 Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

we can read off the formulas for the derivatives $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- Here is a different way of depicting the relationships (not as a tree):



This depicts that f is a function of x, y, and that x and y are both functions of s, t, and u. So we can view f as $f(x(s,t,u), y(s,t,u))$, which means f has derivatives with respect to s, t, and u:

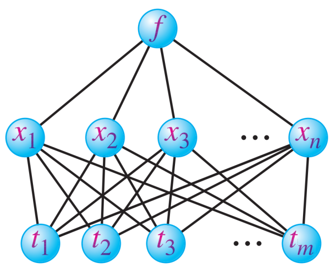
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

- Now for the generalized chain rule:

Theorem (Generalized chain rule)**.** Let $f(x_1, \dots, x_n)$ be a function of n variables. Suppose that each of the variables $x_1, \dots, x_n$ is a differentiable function of m independent variables $t_1, \dots, t_m$.



Then, for $k = 1, \dots, m$,

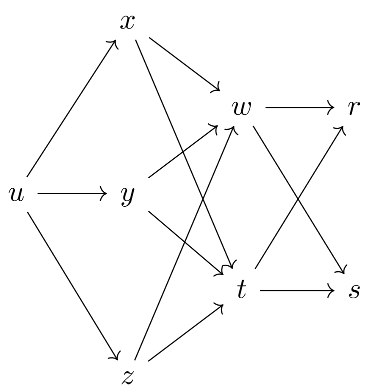
$$\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k}$$

- As expected with chain rule, this can be extended with not just breadth, but depth as well:
  - The Calc 1 version was:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(h(x))) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

  - Here is an example of the Calc 3 version:

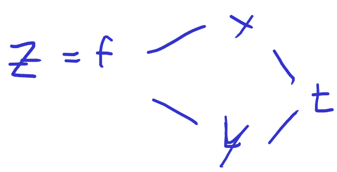
    If $u(x, y, z)$, $x = x(w, t)$, $y = y(w, t)$, $z = z(w, t)$, $w = w(r, s)$, $t = t(r, s)$, depicted as:



Then $\dfrac{\partial u}{\partial s}$ can be computed as an expression involving 6 terms (each term with 3 factors):

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}\frac{\partial t}{\partial s}$$

Example (Similar to Exercise 215)**.** Let $z = f(x, y) = \sqrt{x^2 - y^2}$, $x = x(t) = e^{2t}$, $y = y(t) = e^{-t}$. Calculate $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

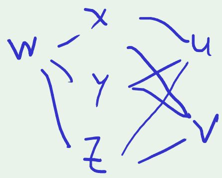


Solution.

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\
&= \frac{2x}{2\sqrt{x^2-y^2}} \cdot 2e^{2t} + \frac{-2y}{2\sqrt{x^2-y^2}} \cdot -e^{-t} \\
&= \boxed{\frac{2xe^{2t}+ye^{-t}}{\sqrt{x^2-y^2}}} \\
&= \boxed{\frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}}
\end{aligned}$$

Example (Similar to Exercise 244). Let

$$\begin{aligned}
w &= f(x,y,z) = 3x^2 - 2xy + 4z^2 \\
x &= x(u,v) = e^u \sin v \\
y &= y(u,v) = e^u \cos v \\
z &= z(u,v) = e^u
\end{aligned}$$



Calculate $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$

Solution.

$$\begin{aligned}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\
&= \boxed{(6x-2y)e^u \sin v + (-2x)e^u \cos v + 8ze^u} \\
&= (6e^u \sin v - 2e^u \cos v)e^u \sin v - 2e^{2u} \sin v \cos v + 8e^{2u} \\
&= e^{2u}\left[6\sin^2 v - 2\sin v\cos v - 2\sin v \cos v + 8\right] \\
&= \boxed{e^{2u}\left[6\sin^2 v - 4\sin v \cos v + 8\right]}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \\
&= \boxed{(6x-2y)e^u \cos v + (-2x)(-e^u \sin v)} \\
&= (6e^u \sin v - 2e^u \cos v)e^u \cos v + 2e^{2u}\sin^2 v \\
&= \boxed{e^{2u}\left[6\sin v\cos v - 2\cos^2 v + 2\sin^2 v\right]}
\end{aligned}$$

3 Implicit differentiation

$y^2 = x^3 + xy$

In Calc 1, you learned the process of implicit differentiation as a process to compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ when y is defined implicitly as a function of x through an equation $f(x, y) = 0$. This method also works for functions of several variables. Suppose that z is defined implicitly as a function of x and y via an equation

$$F(x, y, z) = 0$$

We are unable to solve for $z(x, y)$ explicitly, but we can treat $F(x, y, z)$ as a composite function with x and y as independent variables, and use the Chain Rule to differentiate with respect to x:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\boxed{\frac{\partial z}{\partial x}} = 0$$

We have $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$ (since y does not depend on x). Thus,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = F_x + F_z\frac{\partial z}{\partial x} = 0$$

Solving for $\frac{\partial z}{\partial x}$ gives:

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

Similarly, one can compute

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

Now suppose we have y defined implicitly as a function of x via the equation

$$f(x, y) = 0$$

$y(x)$

Running through the same process, we get:

$$f(x, y) = 0$$
$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}} = 0$$
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f_x}{f_y}}$$

This formula speeds up the process of implicit differentiation learned in Calc 1.

Example (Similar to Exercise 232)**.** Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ if y is defined implicitly by the function

$$f(x,y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$$

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f_x}{f_y} = -\frac{6x-2y+4}{-2x+2y-6} = \boxed{\frac{3x-y+2}{x-y+3}}$$

Calc 1 way:

$$6x - 2(y + xy') + 2yy' + 4 - 6y' = 0$$

$$y'(-2x + 2y - 6) = -6x + 2y - 4$$

$$y' = \frac{-6x+2y-4}{-2x+2y-6} = \frac{3x-y+2}{x-y+3}$$