Contents

1 Directional Derivative

Suppose we want to calculate the rate of change in a direction which isn't x or y, but a linear combination of the two. That is the notion of a directional derivative.

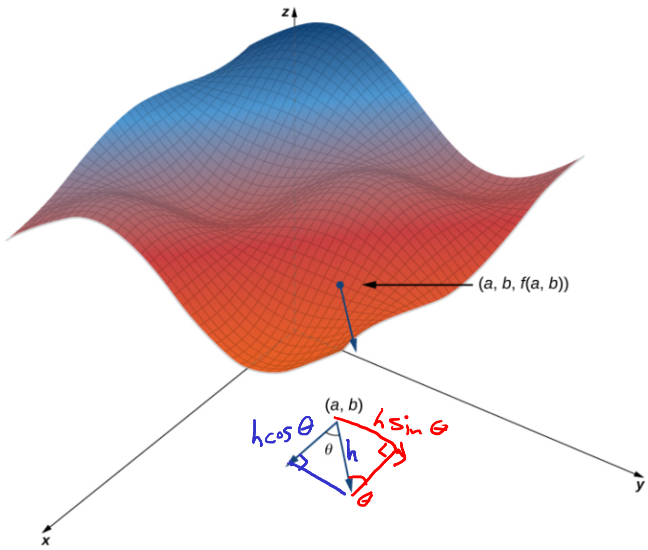

Figure 4.39 Finding the directional derivative at a point on the graph of $z = f(x, y)$. The slope of the black arrow on the graph indicates the value of the directional derivative at that point.

Definition. Given a unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$, the **directional derivative** of f in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(a,b) = \lim_{h\to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

Another way to calculate a directional derivative involves partial derivatives:

Proposition. Let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. The directional derivative of $f(x,y)$ in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= \underbrace{\big\langle f_x(x,y), f_y(x,y) \big\rangle}_{\vec{\nabla} f} \cdot \underset{\vec{u}}{\langle \cos\theta, \sin\theta \rangle} \end{aligned}$$

Proof. Uses chain rule. See the proof of Theorem 4.12 in the book. □

Remark. If the given direction vector $\mathbf{v}$ is *not* a unit vector, you must normalize $\mathbf{v}$ to get the unit vector $\mathbf{u}$, before computing the directional derivative $D_{\mathbf{u}}f$.

2 Gradient

The vector $\langle f_x(x,y), f_y(x,y)\rangle$ has a name, the **gradient** of f, and is denoted $\boldsymbol{\nabla} f$. (The symbol ∇ is called "nabla" or "del")

Definition.

$$\boldsymbol{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$
$$\boldsymbol{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

By abusing notation, we can define (in dimension 3)

$$\boldsymbol{\nabla} := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

from which we get

$$\boldsymbol{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

(This mnemonic will really pay off when we get to divergence and curl, in Section 6.5.)

Formula for directional derivative, using the gradient

$$\boxed{D_{\mathbf{u}} f = \boldsymbol{\nabla} f \cdot \vec{u}}$$
$$\boxed{D_{\mathbf{u}} f(a,b) = \boldsymbol{\nabla} f(a,b) \cdot \vec{u}}$$

Example (Similar to Exercise 281)**.** Calculate $\boldsymbol{\nabla} f(3,-2,4)$, where

$$f(x,y,z) = ze^{2x+3y}$$

Solution. The gradient is

$$\boldsymbol{\nabla} f = \left\langle 2ze^{2x+3y}, 3ze^{2x+3y}, e^{2x+3y} \right\rangle$$

so

$$\boldsymbol{\nabla} f(3,-2,4) = \left\langle 2(4)e^0, 3(4)e^0, e^0 \right\rangle = \boxed{\langle 8, 12, 1\rangle}$$

Example (Similar to Exercise 266, 272)**.** Let $f(x, y) = xe^y$, $P = (2, -1)$, and $\mathbf{v} = \langle 2, 3\rangle$. Calculate the directional derivative in the direction of $\mathbf{v}$.

Solution. First note that $\mathbf{v}$ is not a unit vector. So we replace it with the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, 3\rangle}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

Then

$$\begin{aligned} D_{\mathbf{u}} f(P) &= \nabla f(2, -1) \cdot \mathbf{u} \\ &= \langle e^y, xe^y\rangle \bigg|_{(2,-1)} \cdot \mathbf{u} \\ &= \left\langle e^{-1}, 2e^{-1} \right\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \boxed{\frac{8}{e\sqrt{13}}} \end{aligned}$$

3 Properties of the Gradient

$-1 \le \cos \le 1$

3.1 Gradient and Directional Derivative

Using the dot product–cosine angle formula $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$, we can get

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = \left\|\nabla f(x_0, y_0)\right\| \|\mathbf{u}\| \cos\theta = \left\|\nabla f(x_0, y_0)\right\| \underline{\cos\theta}$$

(θ is the angle between ∇f and $\mathbf{u}$). We can conclude that

- The directional derivative at a point (x_0, y_0) is maximized when $\mathbf{u}$ is pointing in the *same* direction as $\nabla f(x_0, y_0)$.
  - The maximum value of $D_{\mathbf{u}} f(x_0, y_0)$ is $\left\|\nabla f(x_0, y_0)\right\|$
  - Another way to phrase this: The gradient vector $\nabla f(P)$ points in the direction of steepest ascent. This maximum rate of ascent is $\left\|\nabla f(P)\right\|$
- The directional derivative at a point (x_0, y_0) is minimized when $\mathbf{u}$ is pointing in the *opposite* direction as $\nabla f(x_0, y_0)$.
  - The minimum value of $D_{\mathbf{u}} f(x_0, y_0)$ is $-\left\|\nabla f(x_0, y_0)\right\|$

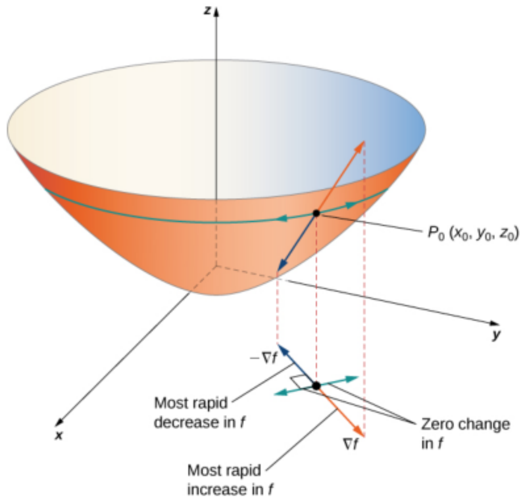

Figure 4.41 The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example (Similar to Exercise 295)**.** Let $f(x,y) = x^4y^{-2}$ and $P = (2,1)$. Find the unit vector that points in the direction of maximum rate of increase at P and determine that maximum rate.

Solution. The gradient points in the direction of maximum rate of increase, so we evaluate the gradient at P:

$$\vec{\nabla} f = \left\langle 4x^3y^{-2}, -2x^4y^{-3} \right\rangle, \quad \vec{\nabla} f(2,1) = \langle 32, -32 \rangle$$

The unit vector in this direction is

$$\mathbf{u} = \frac{\langle 32, -32 \rangle}{\|\langle 32, -32 \rangle\|} = \frac{\langle 32, -32 \rangle}{32\sqrt{2}} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

The maximum rate, which is the rate in this direction, is given by

$$\left\|\vec{\nabla} f(2,1)\right\| = \sqrt{32^2 + (-32)^2} = \boxed{32\sqrt{2}}$$

3.2 Gradient and Level Curves

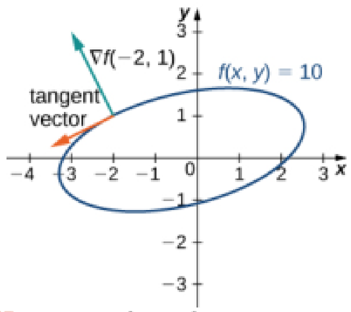

Figure 4.45 Tangent and normal vectors to $2x^2 - 3xy + 8y^2 + 2x - 4y + 4 = 18$ at point $(-2, 1)$.

The gradient is normal to level curves.
This property is the genesis for the Lagrange Multiplier method (Section 4.8).