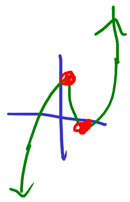
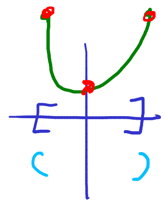
Contents

1 Recalling the Calc 1 version of this topic





- Local extrema
  - Local min (neighboring points all have higher value)
  - Local max (neighboring points all have lower value)
- Absolute extrema
  - Absolute min (a place where the function achieves it minimum value)
  - Absolute max (a place where the function achieves it maximum value)
- Critical points (values x where $f'(x) = 0$ or undefined)
  - "Take the derivative, set equal to zero, and solve"
- Fact: Local extrema occur at critical points.
- Fact: On a closed interval, absolute extrema occur at critical points or endpoints
- There were two tests that could be used to classify the critical point as being a local min / local max / neither.
  - The first derivative test involved looking at $f'(x)$ to the left and right of the critical point $x = p$.
  - The second derivative test invovled plugging the critical point into the second derviative $f''(p)$
    - $f''(p) > 0 \implies$ local min
    - $f''(p) < 0 \implies$ local min

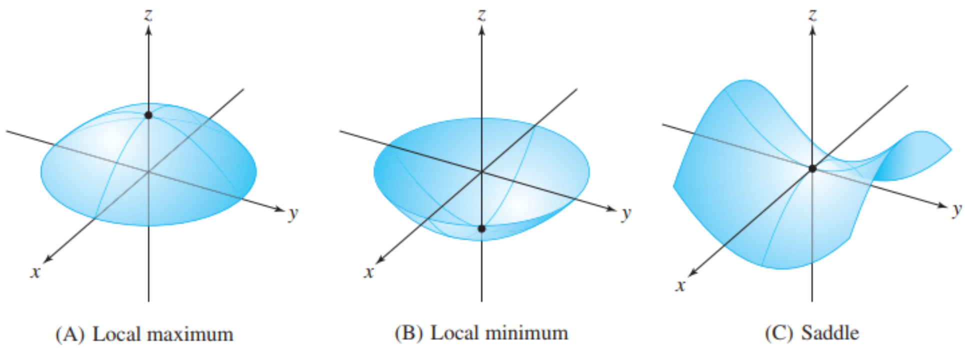
2 Critical points

Definition. A point (x_0, y_0) is a **critical point** for a function f if:

$$\nabla f(x_0, y_0) = \mathbf{0} \text{ or is undefined.}$$

Theorem (Fermat's theorem)**.** Local mins and maxes occur at critical points.

2.1 Types of critical points



2.2 Second Derivative Test

The Second Derivative Test is a way to determine the type of a critical point (a, b) of a function $f(x, y)$. It involves considering the sign of the **discriminant**:

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$$

This is sometimes called the "Hessian determinant". Notice that this formula can be recognized as a determinant:

$$D = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

Theorem (Second Derivative Test)**.** Let $P = (a, b)$ be a critical point for $f(x, y)$. Assume f_{xx}, f_{yy}, f_{xy} are continuous near P. Then:

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then f has a saddle point at (a, b).

(d) If $D = 0$, the test is inconclusive.

Example (Similar to Exercise 312, 320, 331)**.** Find and classify all the critical points for $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

Solution. Start by computing all the first and second order derivatives:

$$\begin{aligned} f_x &= 6xy - 6x & \qquad f_{xx} &= 6y - 6 \\ f_y &= 3x^2 + 3y^2 - 6y & f_{xy} &= 6x \\ & & f_{yy} &= 6y - 6 \end{aligned}$$

Solving for the critical points:

$$\nabla f = \mathbf{0} \implies \begin{cases} f_x = 6xy - 6x = 0 \\ f_y = 3x^2 + 3y^2 - 6y = 0 \end{cases}$$

Factoring a $6x$ from the first equation, we get

$$6x(y - 1) = 0$$

which says that either $x = 0$ or $y = 1$. Trying out each condition on the second equation:

- If $x = 0$:
$$3y^2 - 6y = 3y(y - 2) = 0 \implies y = 0, 2$$
- If $y = 1$:
$$3x^2 + 3 - 6 = 3x^2 - 3 = 3(x^2 - 1) = 0 \implies x = \pm 1$$

This gives 4 critical points:

$$(0, 0), \quad (0, 2), \quad (-1, 1), \quad (1, 1)$$

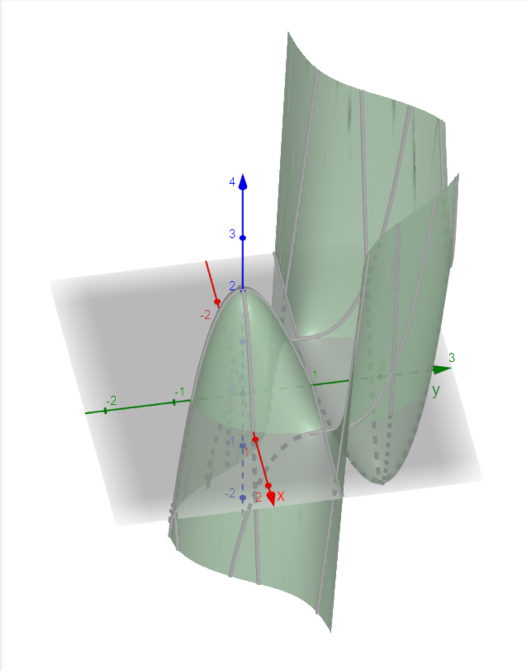
To classify the critical points, we compute the discriminant from the second derivatives:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 36(y - 1)^2 - 36x^2$$

and use the second derivative test on each of the critical points we found:

$$\begin{aligned} &D(0, 0) = 36 > 0 \quad \text{and} \quad f_{xx}(0, 0) = -6 < 0 \\ &D(0, 2) = 36 > 0 \quad \text{and} \quad f_{xx}(0, 2) = 6 > 0 \\ &D(1, 1) = -36 < 0 \\ &D(-1, 1) = -36 < 0 \end{aligned}$$

Thus $(0, 0)$ is a local max, $(0, 2)$ is a local min, and $(1, 1)$ and $(-1, 1)$ are saddle points.



Interactive version: https://www.geogebra.org/calculator/cafnw4fm

3 Absolute Minima and Maxima

Theorem (Extreme Value Theorem)**.** A continuous function $f(x, y)$ on a *closed and bounded* domain D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

So to find the absolute extreme values of a function f on a closed, bounded set D, we check

1. the value of f on the critical points of f in D
2. the values of f on the boundary of D

When we regard f restricted to the boundary of D, this reduces the problem to a 1D absolute extreme values problem (which you've done in Calc 1)

Example (Similar to Exercise 346)**.** Find the absolute extrema of the function

$$f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$$

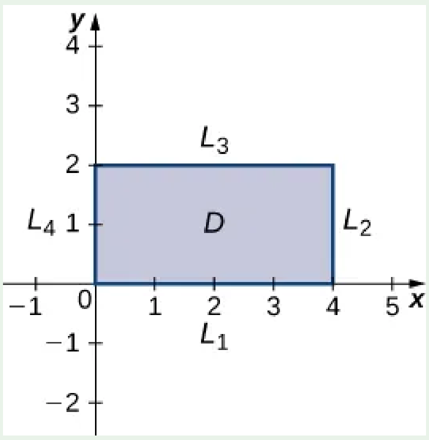
on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$.

Solution.

$$\begin{aligned} f_x &= 2x - 2y - 4 = 0 \implies y = x - 2 \\ f_y &= -2x + 8y - 2 = 0 \implies 6x - 16 - 2 = 0 \implies x = 3 \end{aligned}$$

Thus $(3, 1)$ is a critical point and $f(3, 1) = 17.$

Next, we find extrema of f on the boundary of the domain.



Along L_1:

$$\gamma_1(t) = f(t,0) = t^2 - 4t + 24 \qquad t \in [0,4]$$
$$\gamma_1'(t) = 2t - 4 = 0 \implies t = 2$$

Corresponding point is $(2,0)$. $f(2,0) = 20$.

Along L_2:

$$\gamma_2(t) = f(4,t) = 4t^2 - 10t + 24 \qquad t \in [0,2]$$
$$\gamma_2'(t) = 8t - 10 = 0 \implies t = \frac{5}{4}$$

Corresponding point is $(4, \frac{5}{4})$, and $f(4, \frac{5}{4}) = \frac{71}{4} = 17.75$

Along L_3:

$$\gamma_3(t) = f(t,2) = t^2 - 8t + 36 \qquad t \in [0,4]$$
$$\gamma_3'(t) = 2t - 8 = 0 \implies t = 4$$

$f(4,2) = 20$.

Along L_4:

$$\gamma_4(t) = f(0,t) = 4t^2 - 2t + 24 \qquad t \in [0,2]$$
$$\gamma_4'(t) = 8t - 2 = 0 \implies t = \frac{1}{4}$$

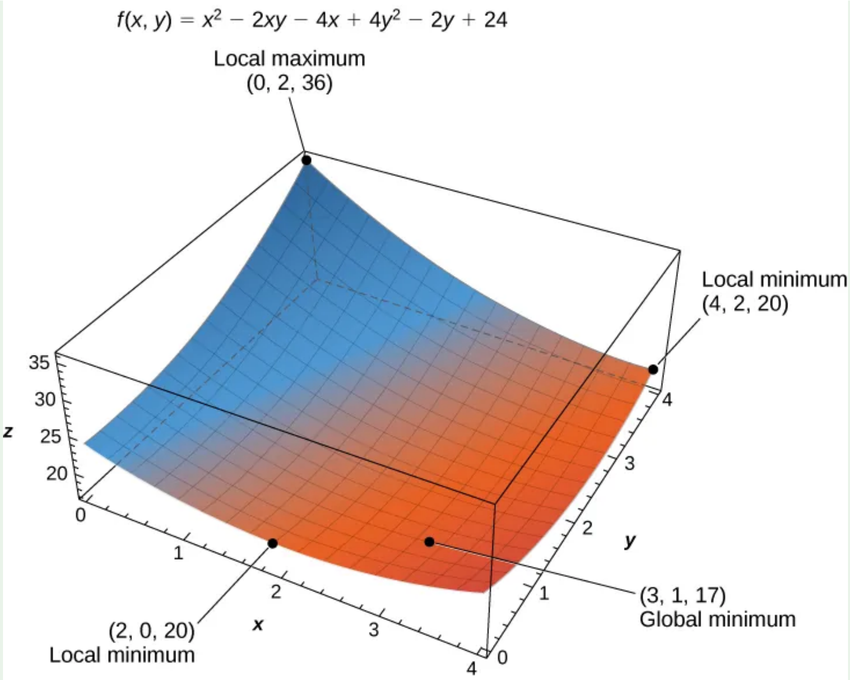
$f(0, \frac{1}{4}) = 23.75$.

Also must test the boundaries of the boundaries (i.e., the corners of the domain):

$$f(0,0) = 24$$
$$f(4,0) = 24$$
$$f(4,2) = 20$$
$$f(0,2) = 36$$

Looking through all the points tested and all the values we obtained,

- the absolute maximum value is 36, occurring at $(0,2)$, and
- the absolute minimum value is 17, occurring at $(3,1)$.



Remark. This is the Web version of Figure 4.53. The PDF version is incorrectly labelled.

Example (OpenStax Calc 3 Example 4.40 b.)**.** Find the absolute extrema of the function

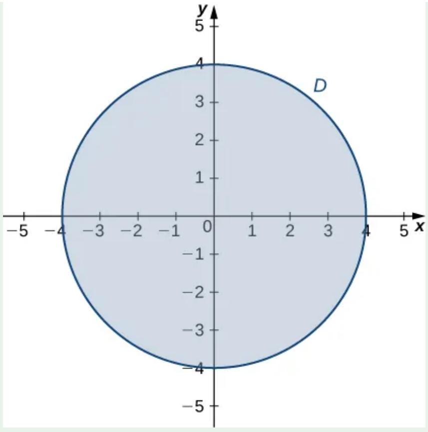
$$g(x, y) = x^2 + y^2 + 4x - 6y$$

on the domain $x^2 + y^2 \leq 16$

Solution.

$$g_x = 2x + 4 = 0 \implies x = -2$$
$$g_y = 2y - 6 = 0 \implies y = 3$$

Therefore $(-2, 3)$ is a critical point of g. $g(-2, 3) = -13$.



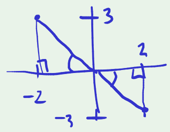
Now for critical points along the boundary. We parameterize the boundary of the circle $x^2 + y^2 = 16$ by

$$x(t) = 4\cos t, \quad y(t) = 4\sin t \qquad t \in [0, 2\pi]$$

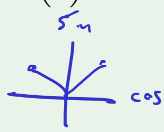
$$g(x, y) = x^2 + y^2 + 4x - 6y$$

$$\begin{aligned}\gamma(t) = g(x(t), y(t)) &= (4\cos t)^2 + (4\sin t)^2 + 4(4\cos t) - 6(4\sin t)\\ &= 16 + 16\cos t - 24\sin t\end{aligned}$$

$$\begin{aligned}\gamma'(t) &= -16\sin t - 24\cos t = 0\\ &\implies -16\sin t = 24\cos t\\ &\implies \tan t = -\frac{3}{2}\end{aligned}$$



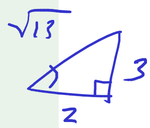
Over the interval $0 \le t \le 2\pi$, this has two solutions: $t = \pi - \arctan\left(\frac{3}{2}\right)$ and $t = 2\pi - \arctan\left(\frac{3}{2}\right)$.



- $t = \pi - \arctan\left(\frac{3}{2}\right)$:

$$\sin t = \sin\left(\pi - \arctan\left(\tfrac{3}{2}\right)\right) = \sin\left(\arctan\left(\tfrac{3}{2}\right)\right) = \frac{3\sqrt{13}}{13}$$

$$\cos t = \cos\left(\pi - \arctan\left(\tfrac{3}{2}\right)\right) = -\cos\left(\arctan\left(\tfrac{3}{2}\right)\right) = -\frac{2\sqrt{13}}{13}.$$



The corresponding boundary critical point is

$$(x(t), y(t)) = (4\cos t, 4\sin t) = \left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right)$$

$$g\left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right) = 16 - 8\sqrt{13} \approx -12.8444$$

- $t = 2\pi - \arctan\left(\frac{3}{2}\right)$:

$$\sin t = \sin\left(2\pi - \arctan\left(\tfrac{3}{2}\right)\right) = -\sin\left(\arctan\left(\tfrac{3}{2}\right)\right) = -\frac{3\sqrt{13}}{13}$$
$$\cos t = \cos\left(2\pi - \arctan\left(\tfrac{3}{2}\right)\right) = \cos\left(\arctan\left(\tfrac{3}{2}\right)\right) = \frac{2\sqrt{13}}{13}.$$

The corresponding boundary critical point is

$$(x(t), y(t)) = (4\cos t, 4\sin t) = \left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$$

$$g\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right) = 16 + 8\sqrt{13} \approx 44.8444$$

Thus the function g has the absolute extreme values:

- Absolute minimum value is -13 at $(-2, 3)$
- Absolute maximum value is $16 + 8\sqrt{13}$ at $(\frac{8}{\sqrt{13}}, -\frac{12}{\sqrt{13}})$

The book's figure for this (Figure 4.55, both PDF and web versions) is labelled incorrectly. Here's a corrected version.