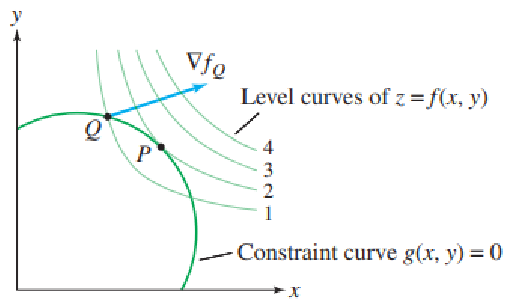
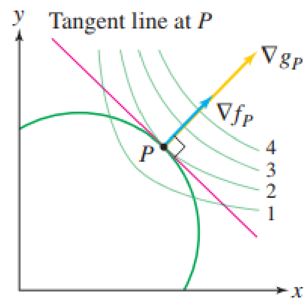
## Contents



Some optimization problems involve finding the extreme values of a function $f(x, y)$ subject to a constraint $g(x, y) = 0$. The method of **Lagrange multipliers** is a general procedure for solving such problems.




(A) $f$ increases as we move to the right along the constraint curve.




(B) The local maximum of $f$ on the constraint curve occurs where $\nabla f_P$ and $\nabla g_P$ are parallel.

**Graphical Insight** Imagine being at point $Q$ in the above figure. We want to increase the value of $f$ while remaining on the constraint curve. The gradient vector $\nabla f(Q)$ points in the direction of maximum increase, but we cannot move in the gradient direction because that would take us off the constraint curve. However, the gradient points to the right, so we can still increase $f$ somewhat by moving to the right along the constraint curve.

We keep moving right until we arrive at the point $P$, where $\nabla f(P)$ is orthogonal to the constraint curve. Once at $P$, we cannot increase $f$ further by moving either to the right or to the left along the constraint curve. Thus $f(P)$ is a local maximum subject to the constraint.

Now, the vector $\nabla g(P)$ is also orthogonal to the constraint curve, so $\nabla f(P)$ and $\nabla g(P)$ must point in the same or opposite direction. In other words,

$$\nabla f(P) = \lambda \nabla g(P)$$

for some scalar $\lambda$ (called a **Lagrange multiplier**). Graphically, this means that a local max subject to the constraint occurs at points $P$ where the level curves of $f$ and $g$ are tangent.

Thus, the strategy to optimize a function $f(x, y)$ subject to a constraint function

$g(x, y) = 0$ is to solve the system of equations

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 0 \end{cases}$$

to find all **constrained critical points**, and evaluate $f$ on those critical points to determine the min/max value.

### Some additional remarks

- The method of Lagrange multipliers is valid in any number of variables.
- The method of Lagrange multipliers can be used where there is more than one constraint equation, but we must add another multiplier for each additional constraint.
  - **Example:** If the problem is to minimize $f(x, y, z)$ subject to constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, then we would solve

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$$

## 1 Hints to the homework problems

$2 \cdot 3 + 2^3$

**#4.359** There are $\boxed{14}$ different constrained critical points. The casework is somewhat tricky. The answers are $\boxed{\pm\frac{2}{\sqrt{3}}}$

**#4.369** This is not too bad. There $\boxed{\text{two}}$ constrained critical points. The answer is $\boxed{2}$

**#4.381** There are $\boxed{\text{two}}$ constrained critical points. Somewhat tricky. Casework is needed. The answer is $\boxed{4 \text{ ft}^3}$

## 2 Simple examples

**Example.** Find the minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$.

$g(x,y,z) = x+y+z-1$

**Solution.** We get the system

$\nabla f = \lambda \nabla g$

$$\begin{cases} \langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle \\ x + y + z = 1 \end{cases} \implies \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x + y + z = 1 \end{cases}$$

Considering the first three equations, we can immediately conclude that $x = y = z$. Using this on the fourth equation, we get $3x = 1 \implies x = \frac{1}{3}$. The critical point on the constaint curve is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Plugging this into $f$ gives the value $3 \cdot \frac{1}{3^2} = \boxed{\frac{1}{3}.}$ To confirm this value is indeed a minimum, we can pick an easy point that satisfies the constraint, such as $(1, 0, 0)$.

**Example.** Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

$g(x,y) = x^2+y^2-136$

**Solution.**

$\nabla f = \lambda \nabla g$

$$\begin{cases} \langle 5, -3 \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 136 \end{cases} \implies \begin{cases} 5 = 2\lambda x \\ -3 = 2\lambda y \\ x^2 + y^2 = 136 \end{cases}$$

Since $\lambda$ cannot be zero (equations 1 and 2 would be invalid), we can solve for $x$ and $y$

$$x = \frac{5}{2\lambda} \qquad y = \frac{-3}{2\lambda}$$

Plugging these into the third equation,

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = \frac{17}{2\lambda^2} = 136$$

Solving for $\lambda$,

$$\lambda^2 = \frac{1}{16} \implies \lambda = \pm\frac{1}{4}$$

If $\lambda = \frac{1}{4}$, we get $x = 10$, $y = -6$.
If $\lambda = -\frac{1}{4}$, we get $x = -10$, $y = 6$.

Plugging these into $f(x, y)$:

$$f(-10, 6) = -68 \qquad f(10, -6) = 68$$

So the minimum is $-68$ and the maximum is $68$.

# 3 Tricky examples

**Example.** Find the maximum and minimum of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 1$. Assume that $x, y, z \geq 0$.

**Solution.**

$$\begin{cases} \langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle \\ \quad x + y + z = 1 \end{cases} \implies \begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x + y + z = 1 \end{cases}$$

Setting the first two equations equal, we get

$$yz = xz \implies (y - x)z = 0$$

so $z = 0$ or $x = y$.

$(xy = 0)$

- Suppose $z = 0$. This makes $\lambda = 0$. Looking at equation 3, this implies either $x = 0$ or $y = 0$.
  - If $x = 0$, to satisfy the constraint, we must have $y = 1$. This gives the point $(0, 1, 0)$
  - If $y = 0$, to satisfy the constraint, we must have $x = 1$. This gives the point $(1, 0, 0)$
- Suppose $y = x$. Then the system reduces to

$$\begin{cases} xz = \lambda \\ x^2 = \lambda \\ 2x + z = 1 \end{cases}$$

Setting the first two equations equal, we get

$$xz = x^2 \implies x(x - z) = 0$$

So either $x = 0$ or $x = z$.

– If $x = 0$, then to satisfy the constraint, $z = 1$. This gives the point $(0, 0, 1)$.
– If $x = z$, then the constraint simplifies to $3x = 1 \implies x = \frac{1}{3}$. This gives the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

We have found four solutions to the inital system of equations. Evaluating $f$ on all of them,

$$f(1,0,0) = f(0,1,0) = f(0,0,1) = 0$$
$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$$

Thus, subject to the constraint, the minimum is 0, the maximum is $\frac{1}{27}$.

**Example** (Similar to Exercise 359). Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the constraint $x + 9y^2 + z^2 = 4$. Assume that $x \geq 0$.

**Solution.** $g(x,y,z) = x + 9y^2 + z^2 - 4$

$$\begin{cases} \langle yz, xz, xy \rangle = \lambda \langle 1, 18y, 2z \rangle \\ x + 9y^2 + z^2 = 4. \end{cases} \implies \begin{cases} yz = \lambda \\ xz = 18y\lambda \\ xy = 2z\lambda \\ x + 9y^2 + z^2 = 4. \end{cases}$$

Start by making the LHS of equations 1–3 the same, multiplying appropriately:

$$\begin{cases} xyz = x\lambda \\ xyz = 18y^2\lambda \\ xyz = 2z^2\lambda \\ x + 9y^2 + z^2 = 4. \end{cases}$$

Setting equations 1 and 2 equal, we get

$$x\lambda = 18y^2\lambda \implies \lambda(x - 18y^2) = 0$$

so either $\lambda = 0$ or $x = 18y^2$.

- If $\lambda = 0$, then notice that the previous system gives that $yz = xz = xy = 0$. This means that at least two of the three variables must be zero. In fact, *exactly* two of the three must be zero—all three being zero would violate the constraint condition. This gives three possibilities:

- If $x = y = 0$, the constraint becomes $z^2 = 4 \implies z = \pm 2$. This gives two points: $(0, 0, \pm 2)$.
- If $x = z = 0$, the constraint becomes $9y^2 = 4 \implies y = \pm\frac{2}{3}$. This gives two points: $(0, \pm\frac{2}{3}, 0)$.
- If $y = z = 0$, the constraint becomes $\cancel{x} = 4$. This gives one point: $(4, 0, 0)$.

- If $x = 18y^2$, the system reduces to

$$\begin{cases} yz = \lambda \\ 18y^2 z = 18y\lambda \\ 18y^3 z = 2z\lambda \\ 27y^2 + z^2 = 4. \end{cases}$$

We further assume $\lambda \neq 0$ as that would fall into the other case. By the first equation, this implies that both $y$ and $z$ are nonzero, so we can divide by them, further reducing our system to

$$\begin{cases} yz = \lambda \\ 9y^3 = \lambda \\ 27y^2 + z^2 = 4. \end{cases}$$

From equations 1, 2, we get

$$yz = 9y^3 \implies y(9y^2 - z) = 0$$

- If $y = 0$, the constraint condition says $z^2 = 4 \implies z = \pm 2$. This gives the points $(0, 0, \pm 2)$ which we already found.
- If $9y^2 = z$, the constraint condition becomes

$X = y^2$

$81x^2 + 27x - 4 = 0$

$$\begin{aligned} 27y^2 + 81y^4 = 4 &\implies 81y^4 + 27y^2 - 4 = 0 \\ &\implies (9y^2 + 4)(9y^2 - 1) = 0 \\ &\implies 9y^2 = 1 \implies y = \pm\frac{1}{3} \end{aligned}$$

This gives two points: $(2, \pm\frac{1}{3}, 1)$

Evaluating at all these points:

$$f(0, 0, \pm 2) = f(0, \pm\frac{2}{3}, 0) = f(4, 0, 0) = 0$$
$$f(2, \pm\frac{1}{3}, 1) = \pm\frac{2}{3}$$

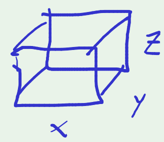
So the absolute minimum is $-\frac{2}{3}$ and the absolute maximum is $\frac{2}{3}$

**Example** (Similar to Exercise 381)**.** Find the dimensions of the box with largest volume if the total surface area is 64 cm$^2$.

**Solution.** Let the box have sides of length $x, y, z$. We want to optimize the volume

$$V(x, y, z) = xyz$$

subject to the constraint




$$SA(x, y, z) = 2xy + 2xz + 2yz = 64$$

Using lagrange multipliers, we get the system

$$\begin{cases} \langle yz, xz, xy\rangle = \lambda \langle 2y + 2z, 2x + 2z, 2x + 2y\rangle \\ 2xy + 2xz + 2yz = 64 \end{cases} \implies \begin{cases} yz = 2\lambda(y + z) \\ xz = 2\lambda(x + z) \\ xy = 2\lambda(x + y) \\ xy + xz + yz = 32 \end{cases}$$

The LHS of equations 1–3 can be made equal by multiplying appropriately:

$$\begin{cases} xyz = 2x\lambda(y + z) \\ xyz = 2y\lambda(x + z) \\ xyz = 2z\lambda(x + y) \\ xy + xz + yz = 32 \end{cases}$$

Comparing the first two equations, we get

$$2x\lambda(y + z) = 2y\lambda(x + z) \implies 2\lambda(xy + xz - xy - yz) = 2\lambda(xz - yz) = 0$$

So either $\lambda = 0$ or $xz = yz$. Note that $\lambda = 0$ is impossible since in the previous system, this would violate the constraint condition. So we must have $xz = yz$. Since $z$ is the side-length of a box, it is nonzero, so we can divide to get $x = y$. Setting equations 2 and 3 equal works similary:

$$\begin{aligned} 2y\lambda(x + z) = 2z\lambda(x + y) &\implies 2\lambda(xy + yz - xz - yz) = 2\lambda(xy - xz) = 0 \\ &\implies xy = xz \\ &\implies y = z \end{aligned}$$

This reduces the constraint condition to

$$3x^2 = 32 \implies x = \pm\sqrt{\frac{32}{3}} = \pm 4\sqrt{\frac{2}{3}} \approx \pm 3.26599$$

Since this is supposed to be a side-length, we only consider the positive solution. Thus the only solution is $x = y = z = 4\sqrt{\frac{2}{3}}$ cm, making the box a cube. It has volume $\frac{128}{3}\sqrt{\frac{2}{3}} \approx 34.8372$ cm$^3$.

To check that this is a maximum, we can compare with another point that satisfies the constraint. Take $x = 2, y = 2$. The constraint then says $4 + 4z = 32 \implies z = 7$. This configuration has volume 28, so we can conclude that the solution we found initially is indeed a maximum.