5.1 Double Integrals over Rectangular Regions

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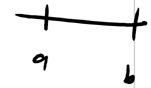


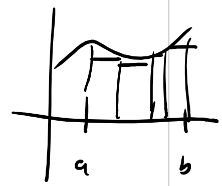
1 §5.1: Double Integrals over Rectangular Regions

Recall that the definition of integral (from Calc 1) was as a Riemann sum,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x$$

where n is the number of subdivision of the interval $[a, b]$, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$

We then introduced the notion of antiderivative and the Fundamental Theorem of Calculus, which allowed us to compute integrals by instead evaluating the antiderivative at the endpoints.



1.1 Volumes and Double Integrals

A rectangular region looks like the following:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$$

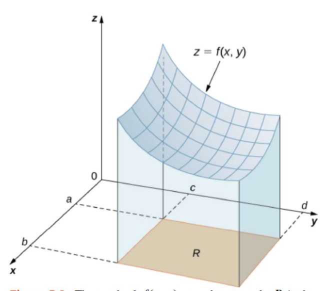
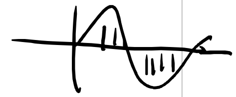


Figure 5.2 The graph of $f(x, y)$ over the rectangle R in the xy-plane is a curved surface.

The **double integral** of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as the limit of a double Riemann sum:

$$\iint_R f(x,y)\,\mathrm{d}A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A.$$



Note: $\mathrm{d}A$ can be expressed as $\mathrm{d}x\,\mathrm{d}y$ or $\mathrm{d}y\,\mathrm{d}x$.

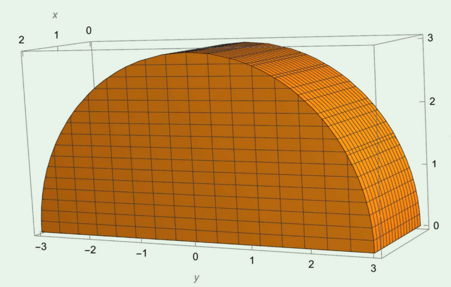
If the function is ever negative, then the double integral can be considered to measure the "signed" volume, just as how the 1D integral from Calc 1 measure the "signed" area.

Example (Similar to Exercise 5.11)**.** The solid lying under the surface $z = \sqrt{9 - y^2}$ and above the rectangular region $R = [0, 2] \times [-3, 3]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)\,\mathrm{d}A$, where $f(x, y) = \sqrt{9 - y^2}$, by finding the volume of the corresponding solid.

$$\int_{-3}^{3}\int_{0}^{2} \sqrt{9-y^2}\,dx\,dy$$

$y = 3\sin\theta$

$$\int_0^2 1\,dx \cdot \underbrace{\int_{-3}^{3} \sqrt{9-y^2}\,dy}$$



Solution. By geometry, the volume of this region is base $\cdot$ height $= \frac{1}{2}\pi(9)(2) = \boxed{9\pi}$

1.2 Properties of Double Integrals

Theorem (Properties of Double Integrals)

Let $f(x, y)$ and $g(x, y)$ be integrable functions over the rectangular region R, let S and T be subregions of R, and let m and M be real numbers.

(i) The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x,y) + g(x,y)]\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}A + \iint_R g(x,y)\,\mathrm{d}A.$$

(ii) If c is a constant, then $cf(x,y)$ is integrable and

$$\iint_R cf(x,y)\,\mathrm{d}A = c\iint_R f(x,y)\,\mathrm{d}A$$

(iii) If $R = S \cup T$ and $S \cap T = \varnothing$ except an overlap on the boundaries, then

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(x,y)\,\mathrm{d}A + \iint_T f(x,y)\,\mathrm{d}A$$

(iv) If $f(x,y) \geq g(x,y)$ for (x,y) in R, then

$$\iint_R f(x,y)\,\mathrm{d}A \geq \iint_R g(x,y)\,\mathrm{d}A$$

(v) If $m \leq f(x,y) \leq M$, then

$$m \times A(R) \leq \iint_R f(x,y)\,\mathrm{d}A \leq M \times A(R)$$

(vi) In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x,y)\,\mathrm{d}A = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right)$$

1.3 Iterated Integrals

The way we evaluate a double integral is to break into two integrals of a single variable, one within the other. These are called **iterated integrals**. The iterated integral for a function $f(x,y)$ over a rectangular region $R = [a,b] \times [c,d]$ can be written as:

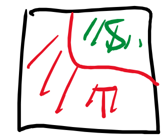
$$\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \left[\overbrace{\int_c^d f(x,y)\,\mathrm{d}y}^{F(x)}\right]\mathrm{d}x$$

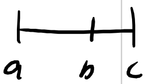
$$\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_a^b f(x,y)\,\mathrm{d}x\right]\mathrm{d}y.$$

The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem.

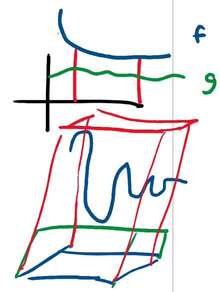
Theorem (Fubini's Theorem)
Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectan-

R





$$\int_a^c = \int_a^b + \int_b^c$$



$$f(x,y) = g(x)h(y)$$

$$\iint f(x,y)\,dA = \int\left(\int g(x)h(y)\,dx\right)dy$$

$$= \int\left(h(y)\cdot \underbrace{\int g(x)\,dx}_{G(x)}\right)dy$$

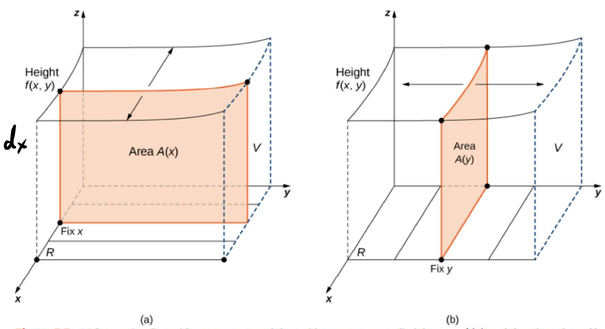
$$= \int g(x)\,dx \cdot \int h(y)\,dy$$

gular region $R = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$. Then we see from Figure 5.7 that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x, y)\,\mathrm{d}A = \iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

$\iint f\, d_y\, d_x$



$\int\int f\, d_x\, d_y$

Figure 5.7 (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

It can actually be beneficial to switch the order of integration to make the computation easier.

Example (Openstax Calc 5.7). Consider the double integral $\iint_R x \sin(xy)\,\mathrm{d}A$ over the region $R = \{(x, y) \mid 0 \le x \le \pi,\ 1 \le y \le 2\}$.

This integral can be expressed in two ways:

$$\int_0^\pi \int_1^2 x \sin(xy)\,\mathrm{d}y\,\mathrm{d}x$$

and

$$\int_1^2 \int_0^\pi x \sin(xy)\,\mathrm{d}x\,\mathrm{d}y$$

$\int x \sin(2x) dx$

Notice that the first is easier to evaluate, because integrating by y first will absorb the x, since $\frac{\partial}{\partial y}(-\cos(xy)) = \sin(xy) \cdot x$.

Using this order, we can compute

$$\begin{aligned}\int_0^\pi \int_1^2 x\sin(xy)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\pi \left[-\cos(xy)\right]_{y=1}^2 \mathrm{d}x \\ &= \cancel{\int_0^\pi \left[-\cos(xy)\right]_{y=1}^2 \mathrm{d}x} \\ &= \int_0^\pi (-\cos 2x + \cos x)\,\mathrm{d}x \\ &= -\frac{1}{2}\sin 2x + \sin x\Big|_0^\pi = 0\end{aligned}$$

If we attempted to integrate using the second order, we would have to use integration by parts, and things would get messy/complicated.

Example (Similar to Exercise 5.19)**.** Calculate the integral by interchanging the order of integration:

$$\int_1^9 \left(\int_4^2 \frac{\sqrt{x}}{y^2}\,\mathrm{d}y\right)\mathrm{d}x$$

Solution.

$$\begin{aligned}\int_1^9 \left(\int_4^2 \frac{\sqrt{x}}{y^2}\,\mathrm{d}y\right)\mathrm{d}x &= \int_4^2 \left(\int_1^9 \frac{\sqrt{x}}{y^2}\,\mathrm{d}x\right)\mathrm{d}y \\ &= \frac{2}{3}\int_4^2 \left[\frac{x^{3/2}}{y^2}\right]_{x=1}^9 \mathrm{d}y \\ &= \frac{2}{3}\cdot 26 \int_4^2 \frac{1}{y^2}\,\mathrm{d}y \\ &= \frac{52}{3}\,\frac{1}{y}\Big|_2^4 = \boxed{-\frac{13}{3}}\end{aligned}$$

$\int a\sqrt{x}\,dx$

$\frac{1}{y^2}[27-1]$

$\int y^{-2}\,dy \quad -\frac{1}{y}]$

$\frac{52}{3}\left(\frac{1}{4}-\frac{1}{2}\right) = \frac{52}{3}\left(-\frac{1}{4}\right)$

Remark. For 5.24,

$$\int_1^e \int_1^e \frac{\sin(\ln x)\cos(\ln y)}{xy}\,\mathrm{d}x\,\mathrm{d}y$$

order does not matter. Can split integral into the form $\int f(y)\,\mathrm{d}y \cdot \int g(x)\,\mathrm{d}x$ and integrate. Answer: $\sin(1)(1-\cos 1)$

$\int \frac{\sin(\ln x)}{x}\,dx$

Remark. For 5.29,

$$\int_0^1 \int_1^2 xe^{x+4y}\,\mathrm{d}y\,\mathrm{d}x$$

The integral can also be split. The integral with respect to x requires IBP (Integration By Parts). Answer: $\frac{1}{4}(e^8 - e^4)$

Example (Similar to Exercise 5.29). Integrate

$$\int_0^3 \int_0^4 xe^{x-3y}\,\mathrm{d}y\,\mathrm{d}x$$

$e^x \cdot e^{-3y}$

Solution.

$$\begin{aligned}
&= \underbrace{\int_0^3 xe^x\,\mathrm{d}x}\cdot \int_0^4 e^{-3y}\,\mathrm{d}y \\
&= \left[(x-1)e^x\right]_0^3 \cdot \left[-\frac{1}{3}e^{-3y}\right]_0^4 \\
&= \left[2e^3 - (-1\cdot 1)\right] \cdot -\frac{1}{3}\left[e^{-12} - 1\right] \\
&= \boxed{-\frac{1}{3}(2e^3+1)(e^{-12}-1)}
\end{aligned}$$

IBP

$\int u\,dv = uv - \int v\,du$

$u =$ | $dv =$

$du =$ | $v =$

	D	I
+	x	e^x
−	1	e^x
+	0	e^x

$\boxed{xe^x - e^x + C}$

1.4 Applications of double integrals

1.4.1 Area

Integrating the constant function 1 over a region R will give the area of R.

$$\text{Area}(R) = \iint_R 1\,\mathrm{d}A$$

$\int_a^b 1\,dx = b - a$

$\iiint_\Omega 1\,dV = \text{vol}(\Omega)$

1.4.2 Average Value

Definition. The **average value** of a function of two variables over a region R is

$$f_{\text{avg}} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,\mathrm{d}A$$

Remark. Compare this with the average value of a function of one variable on an interval $[a, b]$:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Example (Similar to Exercise 5.35). Find the average value of the function over the given rectangles: $f(x, y) = x^2 - y^3$ over $R = [0, 2] \times [0, 3]$

Solution.

$$
\begin{aligned}
f_{\text{avg}} &= \frac{1}{6}\iint_R (x^2 - y^3)\,\mathrm{d}A \\
&= \frac{1}{6}\int_0^3 \int_0^2 (x^2 - y^3)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{6}\int_0^3 \left[\frac{x^3}{3} - xy^3\right]_{x=\cancel{2}\,0}^{\cancel{3}\,2} \mathrm{d}y \\
&= \frac{1}{6}\int_0^3 \left(\frac{8}{3} - 2y^3\right)\mathrm{d}y \\
&= \frac{1}{6}\left[\frac{8}{3}y - \frac{y^4}{2}\right]_0^3 \\
&= \frac{1}{6}\left(8 - \frac{81}{2}\right) = \boxed{-\frac{65}{12}}
\end{aligned}
$$