5.2: Double Integrals over General Regions

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1 §5.2: Double Integrals over General Regions

1.1 General Regions of Integration

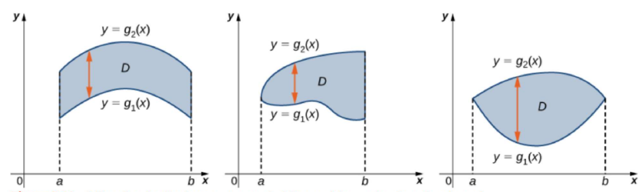


Figure 5.13 A Type I region lies between two vertical lines and the graphs of two functions of x.

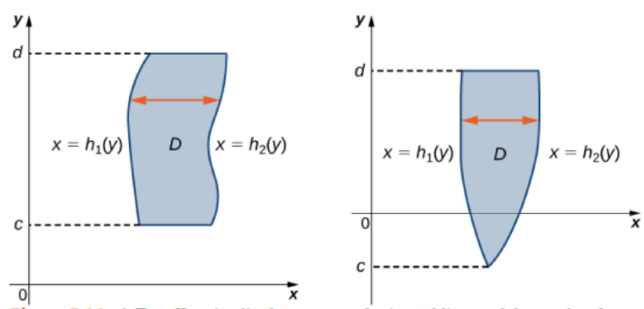


Figure 5.14 A Type II region lies between two horizontal lines and the graphs of two functions of y.

Regions of type I and II

Definition. A region D in the xy-plane is of **Type I** if it lies between two *vertical* lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. Expressed symbolically:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

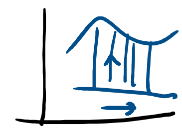
A region D in the xy-plane is of **Type II** if it lies between two *horizontal* lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. Expressed symbolically:

$$D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

1.2 Double Integrals over Nonrectangular Regions

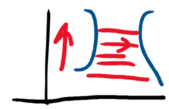
Theorem (Fubini's Theorem)
For a function $f(x, y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x, y)\,\mathrm{d}A = \iint_D f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\right] \mathrm{d}x.$$



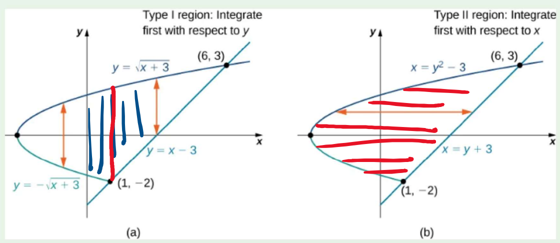
Similarly, for a function $f(x, y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x, y)\,\mathrm{d}A = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\,\mathrm{d}x\right] \mathrm{d}y.$$



Example (Openstax Calc 3 Example 5.13). Evaluate the integral $\iint_D (3x^2 + y^2)\,\mathrm{d}A$ where $D = \{(x, y) \mid -2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

dydx



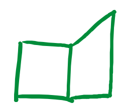
dx dy

Solution. It is easier to integrate this as a Type II region.

$$\begin{aligned}\iint_D (3x^2+y^2)\,\mathrm{d}A &= \int_{y=-2}^{3}\int_{y^2-3}^{y+3}(3x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-2}^{3}(x^3+xy^2)\Big|_{x=y^2-3}^{y+3}\,\mathrm{d}y \\ &= \int_{-2}^{3}(-y^6+8y^4+2y^3-12y^2+27y+54)\,\mathrm{d}y \\ &= -\frac{y^7}{7}+\frac{8y^5}{5}+\frac{y^4}{2}-4y^3+\frac{27y^2}{2}+54y\Big|_{-2}^{3} = \boxed{\frac{2375}{7}}\end{aligned}$$

Theorem (Decomposing Regions into Smaller Regions)
Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$



1.3 Changing the Order of Integration

Sometimes one order of integraiton is significantly simpler than the other.

Example (Openstax Calc 3 Example 5.15). Compute

$$\int_0^{\sqrt{2}}\int_0^{2-x^2} xe^{x^2}\,\mathrm{d}y\,\mathrm{d}x$$

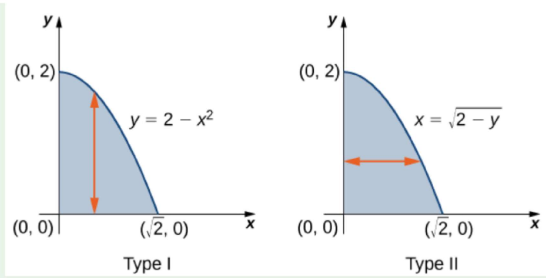
$$= \int_0^{\sqrt{2}} \left[xe^{x^2}\cdot y\right]_{y=0}^{2-x^2} dx$$

$$\int_0^{\sqrt{2}} x(2-x^2)e^{x^2}\,dx$$

Solution. In the order presented, this integral is challenging. Doing the inner integral gives the result

$$\int_0^{\sqrt{2}}(2x-x^3)e^{x^2}\,\mathrm{d}x$$

which although doable, is a bit bothersome. If instead, we swap the order of integration,



$\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C$

we get

$$\begin{aligned}
\int_0^{\sqrt{2}} \int_0^{2-x^2} x e^{x^2} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^2 \int_0^{\sqrt{2-y}} x e^{x^2} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^2 \left[\frac{1}{2} e^{x^2} \Big|_{x=0}^{\sqrt{2-y}} \right] \mathrm{d}y \\
&= \int_0^2 \frac{1}{2} \left(e^{2-y} - 1 \right) \mathrm{d}y \\
&= -\frac{1}{2} \left(e^{2-y} + y \right) \Big|_0^2 \\
&= \boxed{-\frac{1}{2} \left(3 - e^2 \right)}
\end{aligned}$$

1.4 Calculating Volumes, Areas, and Average Values

Area of a plane-bounded region:

$$\iint_D 1 \, \mathrm{d}A$$

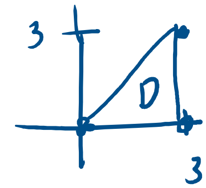
Average value of a function over a general region:

$$f_{\text{avg}} = \frac{1}{A(D)} \iint_D f(x, y) \, \mathrm{d}A$$

where $A(D) = \iint_D 1 \, \mathrm{d}A$.

1.5 Similar problems

Example (Similar to Exercise 5.78)**.** Evaluate the double integral $\iint_D f(x, y) \, \mathrm{d}A$ over the region D:
$f(x, y) = \cos x$ and D is the triangular region with vertices $(0, 0)$, $(3, 0)$, and $(3, 3)$.



Solution. Two setups:

$$\int_0^3 \int_0^x \cos x \, \mathrm{d}y \, \mathrm{d}x$$

$= \int_0^3 [\cos x \cdot y]_0^x \, dx$

$= \int_0^3 x \cos x \, dx$

and

$$\int_0^3 \int_y^3 \cos x \, \mathrm{d}x \, \mathrm{d}y$$

D	I
+ x	$\cos x$
− 1	$\sin x$
+ 0	$-\cos x$

$= x \sin x + \cos x \Big]_0^3$

$= 3 \sin x + \cos 3 - (0+1)$

The latter one is easier (avoids IBP):

$$\begin{aligned}
\int_0^3 \int_y^3 \cos x \, \mathrm{d}x \, \mathrm{d}y &= \int_0^3 [\sin x]_{x=y}^3 \, \mathrm{d}y \\
&= \int_0^3 (\sin 3 - \sin y) \, \mathrm{d}y \\
&= \sin 3 \cdot y + \cos y \Big|_0^3 \\
&= \boxed{3 \sin 3 + \cos 3 - 1}
\end{aligned}$$

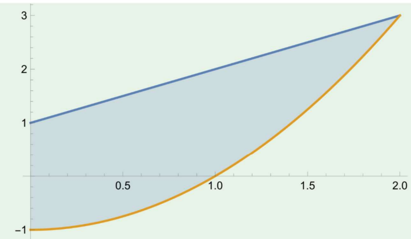
Example (Similar to Exercise 5.80)**.** Evaluate the iterated integral

$$\int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3) \, \mathrm{d}y \, \mathrm{d}x$$

Solution.

$$\begin{aligned}
\int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3) \, \mathrm{d}y \, \mathrm{d}x &= \int_0^1 \left[2xy^2 - \frac{1}{4} y^4 \right]_{y=x^3}^{\sqrt{x}} \mathrm{d}x \\
&= \int_0^1 \left(\frac{7}{4} x^2 - 2x^7 + \frac{1}{4} x^{12} \right) \mathrm{d}x \\
&= \left[\frac{7}{12} x^3 - \frac{1}{4} x^8 + \frac{1}{52} x^{13} \right]_0^1 = \boxed{\frac{55}{156}}
\end{aligned}$$

Example (Similar to Exercise 5.90)**.** The region D bounded by $x = 0$, $y = x + 1$, and $y = x^2 - 1$ is shown in the following figure. Find the area $A(D)$ of the region D.

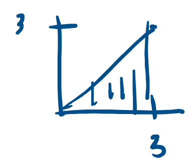


Solution.

$$\begin{aligned} A(D) &= \iint_D 1\,\mathrm{d}A \\ &= \int_0^2 \int_{x^2-1}^{x+1} 1\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^2 \Big[y\Big]_{y=x^2-1}^{x+1} \mathrm{d}x \\ &= \int_0^2 (x+1-x^2+1)\,\mathrm{d}x = \int_0^2 (-x^2+x+2)\,\mathrm{d}x \\ &= -\frac{1}{3}x^3+\frac{1}{2}x^2+2x\Big|_0^2 = \boxed{\frac{10}{3}} \end{aligned}$$

Remark. If we wanted to integrate in the order $\mathrm{d}x\,\mathrm{d}y$, this region would need to be split into two separate regions (the bottom function changes at $y=1$)

Example (Similar to Exercise 5.94). Find the average value of the function $f(x,y) = \cos x$ over the triangular region with vertices $(0,0)$, $(3,0)$, $(3,3)$. (Same setup as my similar example to 5.78.)

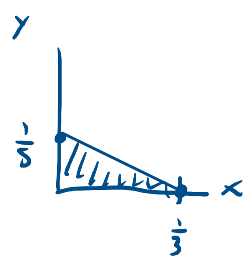


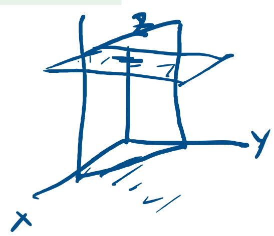
Solution. The area of the region (by geometry) is $\frac{9}{2}$. So

$$f_{\text{avg}} = \boxed{\frac{2}{9}(3\sin 3 + \cos 3 - 1)}$$

$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$

Example (Similar to Exercise 5.109). Find the volume of the solid situated in the first octant and bounded by the planes $3x+5y=1$, $x=0$, $y=0$, $z=10$, and $z=0$.





Solution. The solid is a triangular prism. The volume is $\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{3} \cdot 10 = \boxed{\frac{1}{3}}$.