Contents

1 §5.3: Double Integrals in Polar Coordinates

1.1 Polar Rectangular Regions of Integration

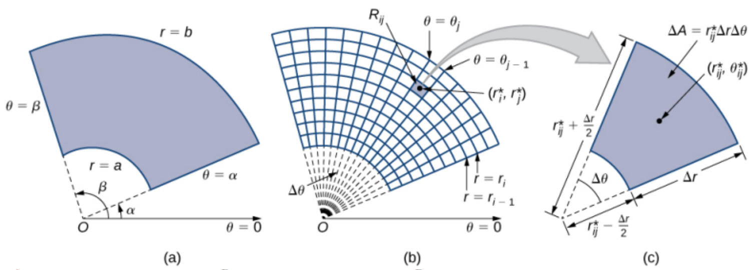


Figure 5.28 (a) A polar rectangle R (b) divided into subrectangles R_{ij}. (c) Close-up of a subrectangle.

A polar rectangle is of the form

$$R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

A double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates.

Jacobian

$$\iint_R f(r,\theta)\,\mathrm{d}A = \iint_R f(r,\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta = \int_\alpha^\beta \int_a^b f(r,\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Notice that $\mathrm{d}A$ is replaced by $r\,\mathrm{d}r\,\mathrm{d}\theta$ when working in polar coordinates.

1.2 General Polar Regions of Integration

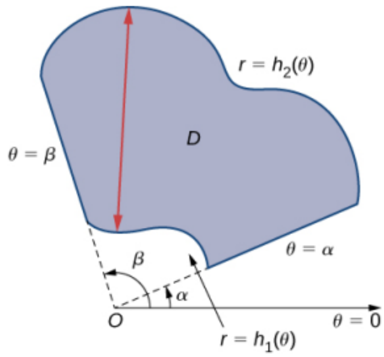


Figure 5.32 A general polar region between $\alpha < \theta < \beta$ and $h_1(\theta) < r < h_2(\theta)$.

A general polar region is described as

$$R = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

A double integral over the above general polar region D is written as:

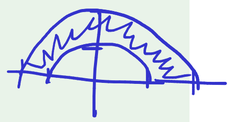
$$\iint_D f(r,\theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta$$

1.3 Examples

$x = r\cos\theta$
$y = r\sin\theta$

Example (Openstax Calc 3 Example 5.25)**.** Evaluate the integral $\iint_R 3x\,\mathrm{d}A$ over the region $R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

Solution.

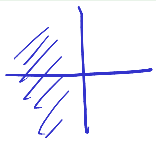


$$\begin{aligned}\iint_R 3x\,\mathrm{d}A &= \int_0^\pi \int_1^2 3r\cos\theta\ r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^\pi \cos\theta\,\mathrm{d}\theta \cdot \int_1^2 3r^2\,\mathrm{d}r \\ &= [\sin\theta]_0^\pi \cdot \left[\cancel{\tfrac{1}{3}}r^3\right]_1^2 = 0\end{aligned}$$

Example (Openstax Calc 3 Example 5.27)**.** Evaluate the integral $\iint_R (x+y)\,\mathrm{d}A$ where $R = \{(x,y) \mid 1 \le x^2 + y^2 \le 4,\ x \le 0\}$

$1 \le r^2 \le 4$

$1 \le r \le 2$



Solution. The region to integrate over can be expressed in polar as

$$R = \{(r, \theta) \mid 1 \leq r \leq 2,\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$$

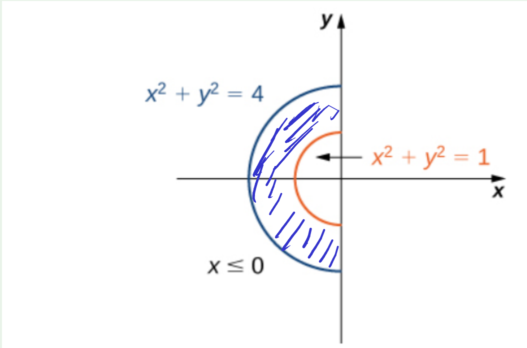


Figure 5.31 The annular region of integration R.

The problem can thus be converted to polar:

$$\begin{aligned}
\iint_R (x+y)\,\mathrm{d}A &= \int_{\pi/2}^{3\pi/2} \int_1^2 (r\cos\theta + r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \left(\int_1^2 r^2\,\mathrm{d}r\right)\left(\int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta)\,\mathrm{d}\theta\right) \\
&= \left[\frac{r^3}{3}\right]_1^2 \cdot [\sin\theta - \cos\theta]_{\pi/2}^{3\pi/2} \\
&= \frac{7}{3} \cdot -2 = \boxed{-\frac{14}{3}}
\end{aligned}$$

sin

cos

$\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}$

$-1 \quad -1$

Example (Openstax Calc 3 Example 5.28)**.** Evaluate the integral $\iint_D r^2 \sin\theta\, r\,\mathrm{d}r\,\mathrm{d}\theta$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution. The region D can be described as $\{(r,\theta) \mid 0 \leq \theta \leq \pi,\ 0 \leq r \leq 1 + \cos\theta\}$

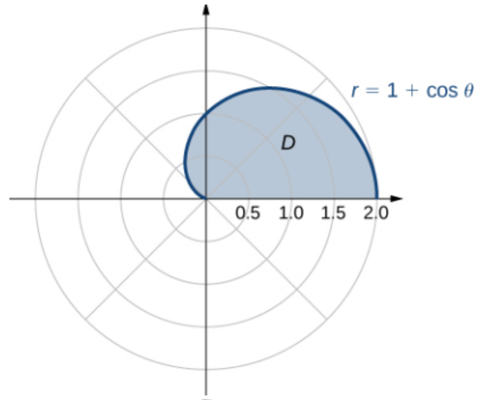


Figure 5.33 The region D is the top half of a cardioid.

The integral can be computed as

$$\begin{aligned}
\iint_D r^2 \sin\theta \; r \, \mathrm{d}r \, \mathrm{d}\theta &= \int_0^{\pi} \int_0^{1+\cos\theta} r^3 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \frac{1}{4} \int_0^{\pi} \left[r^4 \right]_{r=0}^{1+\cos\theta} \sin\theta \, \mathrm{d}\theta \\
&= \frac{1}{4} \int_0^{\pi} (1+\cos\theta)^4 \sin\theta \, \mathrm{d}\theta \\
&= -\frac{1}{4} \left[\frac{(1+\cos\theta)^5}{5} \right]_0^{\pi} \\
&= \boxed{\frac{8}{5}}
\end{aligned}$$

$u = 1 + \cos\theta$

$4 \quad \pi$

$- \; - \frac{2^5}{4 \cdot 5}$

Remark. Note that the iterated integral cannot be split into a product of two integrals, since the bounds couple the two integrals together (the region is *not* polar rectangular).

Example (Gaussian integral)**.** Determine

$$\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x$$

Solution. Let $I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$. Then

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\cdot r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi\cdot -\frac{1}{2}\left[e^{-r^2}\right]_0^{\infty} \\
&= -\pi(0-1) = \pi \\
\implies I &= \boxed{\sqrt{\pi}}
\end{aligned}$$

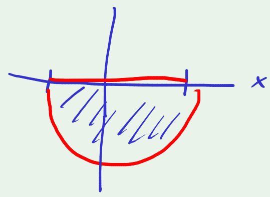
$u = -r^2$

$du = -2r\,dr$

$\frac{1}{e^{r^2}}$

Example (Similar to Exercise 5.148)**.** Evaluate the following integral by first converting to polar:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \cos\left(x^2+y^2\right)\mathrm{d}y\,\mathrm{d}x$$



Solution. The region in polar is

$$\{(r,\theta) \mid 0 \le r \le 1,\ \pi \le \theta \le 2\pi\}$$

The integral becomes

$$\begin{aligned}
&= \int_{\pi}^{2\pi}\int_{0}^{1} \cos\left(r^2\right) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \pi\left[\frac{1}{2}\sin\left(r^2\right)\right]_{r=0}^{1} \\
&= \boxed{\frac{\pi}{2}\sin(1)}
\end{aligned}$$

$u = r^2$

$du = 2r\,dr$