Contents

1 §5.4: Triple Integrals

Theorem 1 (Fubini's Theorem for Triple Integrals)
If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_e^f \int_c^d \int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

Example (Similar to Exercise 5.182)**.** Evaluate the triple integral

$$\int_0^1 \int_2^4 \int_{-1}^5 (x + yz^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Solution.

$$\begin{aligned}
&= \int_0^1 \int_2^4 \left[\frac{x^2}{2} + xyz^2\right]_{x=-1}^5 \mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 \int_2^4 (12 + 6yz^2)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 \left[12y + 3y^2z^2\right]_{y=2}^4 \mathrm{d}z \\
&= \int_0^1 (24 + 36z^2)\,\mathrm{d}z \\
&= \left[24z + 12z^3\right]_{z=0}^1 = \boxed{36}
\end{aligned}$$

Example (Similar to Exercise 5.186)**.** Evaluate the triple integral $\iiint_B x^2yz\,\mathrm{d}V$ where $B = \{(x, y, z) \mid -2 \le x \le 1, 0 \le y \le 3, 1 \le z \le 5\}$ Use the integration order z then x then y.

Solution. Since the domain is a rectangular box, the bounds for the integrals can be easily read off from the domain:

$$\begin{aligned}\iiint_B x^2yz\,\mathrm{d}V &= \int_0^3 \int_{-2}^1 \int_1^5 x^2yz\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^3 \int_{-2}^1 x^2y\left[\frac{z^2}{2}\right]_{z=1}^5 \mathrm{d}x\,\mathrm{d}y \\ &= \int_0^3 \int_{-2}^1 12x^2y\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^3 \left[4x^3y\right]_{x=-2}^1 \mathrm{d}y \\ &= \int_0^3 36y\,\mathrm{d}y = \left[18y^2\right]_0^3 = \boxed{162}\end{aligned}$$

Remark. If the problem hadn't given a specific integration order to use, I would have instead split this into a product of three integrals:

$$\begin{aligned}\iiint_B x^2yz\,\mathrm{d}V &= \int_{-2}^1 x^2\,\mathrm{d}x \cdot \int_0^3 y\,\mathrm{d}y \cdot \int_1^5 z\,\mathrm{d}z \\ &= \left[\frac{x^3}{3}\right]_{-2}^1 \cdot \left[\frac{y^2}{2}\right]_0^3 \cdot \left[\frac{z^2}{2}\right]_1^5 \\ &= 3 \cdot \frac{9}{2} \cdot 12 = \boxed{162}\end{aligned}$$

1.1 Triple Integrals over a General Bounded Region

Now we consider more general bounded regions:

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

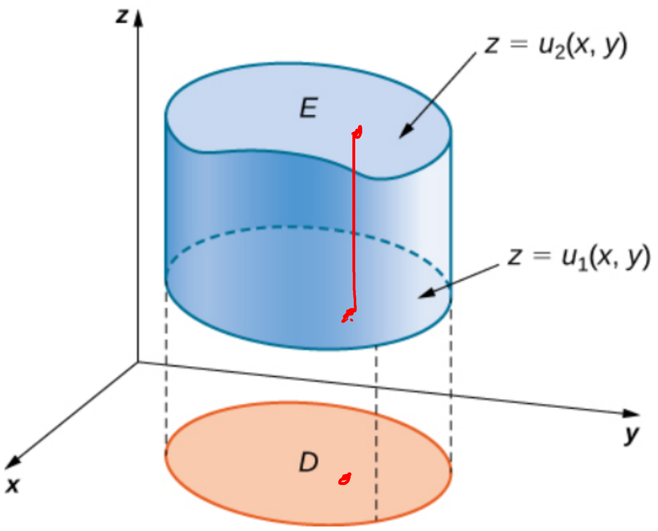


Figure 5.42 We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection D of E onto the xy-plane.

Theorem 2 (Triple Integral over a General Region)
The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \left\{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\right\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, \mathrm{d}z\right] \mathrm{d}A.$$

Things are similar for

$$E = \left\{(x, y, z) \mid (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\right\}$$

and

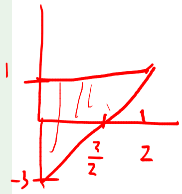
$$E = \left\{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\right\}$$

Example (Similar to Exercise 5.191)**.** Evaluate the triple integral $\iiint_E (3x - 4y + 5z)\,\mathrm{d}V$ over the bounded region

$$E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 2x - 3, 0 \le z \le 1\}$$

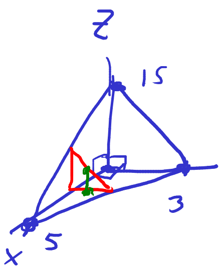
Orders dy dx dz

dz dy dx also work



Solution.

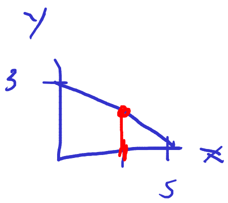
$$\begin{aligned}
&= \int_0^2 \int_0^1 \int_0^{2x-3} (3x - 4y + 5z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_0^2 \int_0^1 \Big[3xy - 2y^2 + 5yz\Big]_{y=0}^{2x-3} \mathrm{d}z\,\mathrm{d}x \\
&= \int_0^2 \int_0^1 (3x(2x-3) - 2(2x-3)^2 + 5(2x-3)z)\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_0^2 \int_0^1 (6x^2 - 9x - 8x^2 + 24x - 18 + 10xz - 15z)\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_0^2 \int_0^1 (-2x^2 + 10xz + 15x - 15z - 18)\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_0^2 (-2x^2 + 5x + 15x - \frac{15}{2} - 18)\,\mathrm{d}x \\
&= \int_0^2 (-2x^2 + 20x - \frac{51}{2})\,\mathrm{d}x \\
&= \boxed{-\frac{49}{3}}
\end{aligned}$$



Example (Similar to Exercise 5.231)**.** Find the volume of the solid E bounded by $z = 15 - 3x - 5y$ in the first octant.

Solution. This is a triangular pyramid, with vertices at $(0, 0, 15)$, $(5, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 0)$. Using geometry, we know that the volume of a pyramid is $\frac{1}{3}Bh = \frac{1}{3}(\frac{1}{2}(5)(3))(15) = \boxed{\frac{75}{2}}$.

But we should probably do this using integrals. Note that along the xy-plane, $x \in [0, 5]$ and $y \in [0, \frac{15-3x}{5}]$. Once these two are fixed, then $z \in [0, 15 - 3x - 5y]$.



$3x + 5y = 15$

$5y = 15 - 3x$

$y = \frac{15 - 3x}{5}$

This gives the integral

$$\begin{aligned} V &= \int_0^5 \int_0^{\frac{15-3x}{5}} \int_0^{15-3x-5y} 1 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^5 \int_0^{\frac{15-3x}{5}} (15 - 3x - 5y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^5 \left[(15-3x)y - \frac{5}{2} y^2 \right]_{y=0}^{15-3x/5} \mathrm{d}x \\ &= \int_0^5 \frac{1}{10} (15-3x)^2 \, \mathrm{d}x \\ &= \frac{1}{30} \frac{-1}{3} \left. (15-3x)^3 \right|_0^5 \\ &= \boxed{\frac{75}{2}} \end{aligned}$$