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1 §5.5: Triple Integrals in Cylindrical and Spherical Coordinates

The point of this section is that sometimes, an integral is easier/more natural to compute in a different coordinate system (cylindrical or spherical, rather than cartesian). This is typically due to the domain of the integral being easier to express in the other coordinate system, though sometimes (as in the Gaussian integral) it can be due to the integrand simplifying after changing coordinates.

Both play a role, as one has to convert both the domain and the integrand over the other coordinate system. And don't forget the Jacobian!

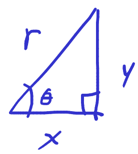
Cyl: r

sph: $\rho^2 \sin \varphi$

1.1 Review of Cylindrical Coordinates

Conversion formulas

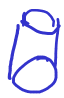
$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$
$$z = z \qquad z = z$$

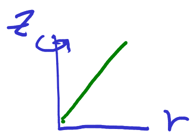


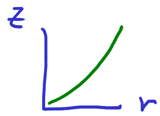
1.2 Integration in Cylindrical Coordinates

Equations of some common shapes No memorization required. These can be attained from the conversion formulas.

	Rectangular	Cylindrical
Circular cylinder	$x^2 + y^2 = c^2$	$r = c$
Circular cone	$z^2 = c^2(x^2 + y^2)$	$z = cr$
Sphere	$x^2 + y^2 + z^2 = c^2$	$r^2 + z^2 = c^2$
Paraboloid	$z = c(x^2 + y^2)$	$z = cr^2$







Fubini's Theorem

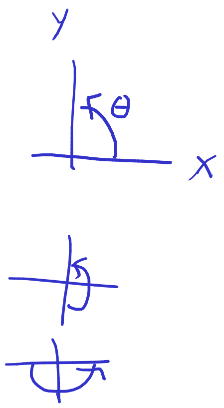
Theorem 1 (Fubini's Theorem in Cylindrical Coordinates)
Suppose that $g(x, y, z)$ is continuous on a cylindrical box

$$B = \{(r, \theta, z) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\}.$$

Then $g(x, y, z) = g(r\cos\theta, r\sin\theta, z) = f(r, \theta, z)$ and

$$\iiint_B g(x, y, z)\,\mathrm{d}V = \int_c^d \int_\alpha^\beta \int_a^b f(r, \theta, z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

1.3 Review of Spherical Coordinates



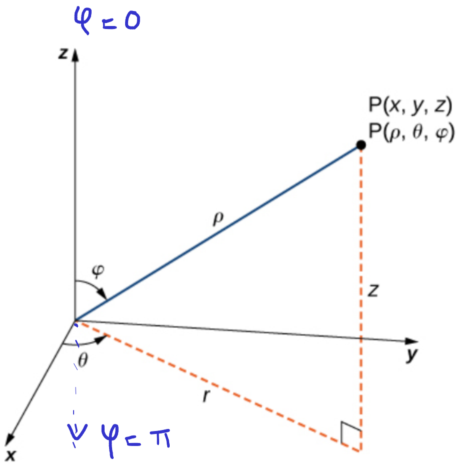
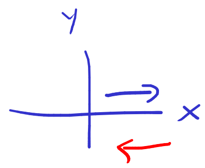
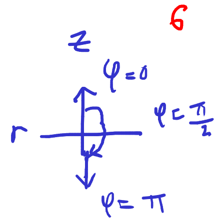


Figure 5.55 The spherical coordinate system locates points with two angles and a distance from the origin.



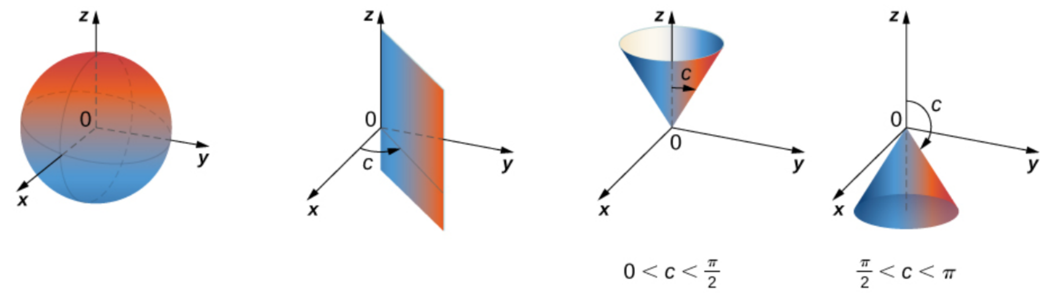
$\int_6^2 = -\int_2^6$



Conversion formulas

$$x = \rho\sin\varphi\cos\theta \qquad \rho^2 = x^2 + y^2 + z^2$$
$$y = \rho\sin\varphi\sin\theta \qquad \tan\theta = \frac{y}{x}$$
$$z = \rho\cos\varphi \qquad \varphi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

Regions in spherical A few solid regions that are convenient to express in spherical coordiantes:



Sphere $\rho = c$ (constant) Half plane $\theta = c$ (constant) Half cone $\varphi = c$ (constant)

Figure 5.56 Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

1.4 Integration in Spherical Coordinates

A **spherical box** is of the form

$$B = \{(\rho, \theta, \varphi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$$

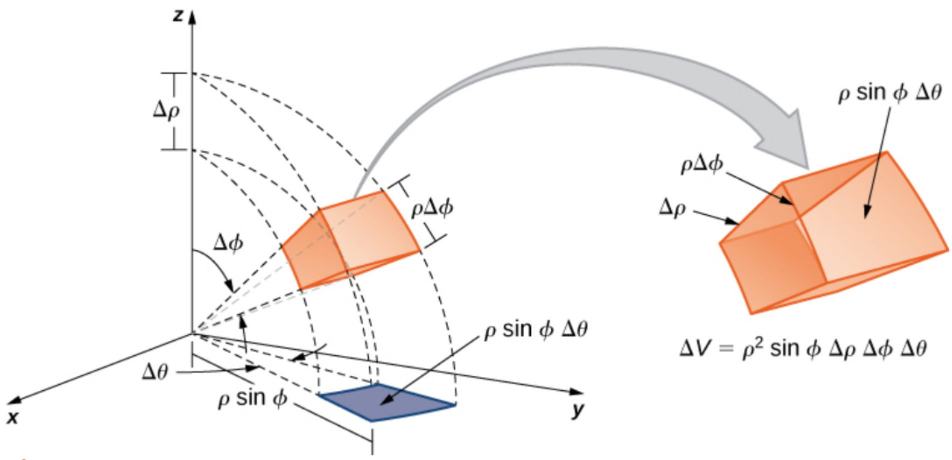


Figure 5.57 The volume element of a box in spherical coordinates.

Theorem 2 (Fubini's Theorem for Spherical Coordinates)
If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_\alpha^\beta \int_\gamma^\psi \int_a^b f(\rho, \theta, \varphi)\underbrace{\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}_{dV}.$$

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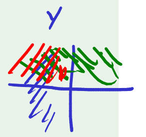
$$\iiint_B f \, dV$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

1.5 Examples Similar to the HW

Example (Similar to Exercise 5.241). Evaluate the triple integral $\iiint_E f(x,y,z)\,\mathrm{d}V$ over the solid E, where $f(x,y,z) = xz$, and

$$E\ \cancel{B} = \{(x,y,z) \mid x^2+y^2 \le 16,\ x \le 0,\ y \ge 0,\ 0 \le z \le 1\}$$



$r^2 \le 16$

$0 \le r \le 4$

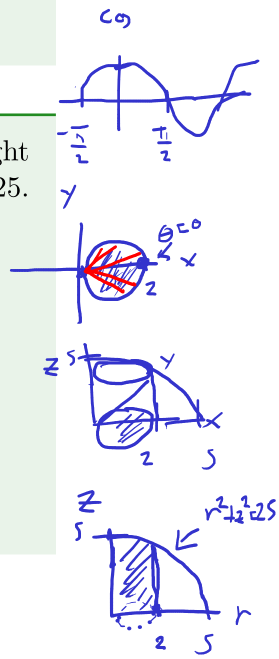
$\theta \in [\frac{\pi}{2}, \pi]$

Solution. Converting the function to cylindrical: $f = r\cos(\theta)z$
Converting the region to cylindrical:

$$E\ \cancel{B} = \{(r,\theta,z) \mid 0 \le r \le 4,\ \frac{\pi}{2} \le \theta \le \pi,\ 0 \le z \le 1\}$$

Integrating,

$$\begin{aligned}
\iiint_E f\,\mathrm{d}V &= \int_0^1 \int_{\pi/2}^{\pi} \int_0^4 r\cos(\theta)z \cdot r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_0^4 r^2\,\mathrm{d}r \cdot \int_{\pi/2}^{\pi} \cos(\theta)\,\mathrm{d}\theta \cdot \int_0^1 z\,\mathrm{d}z \\
&= \left[\frac{r^3}{3}\right]_0^4 \cdot [\sin(\theta)]_{\pi/2}^{\pi} \cdot \left[\frac{z^2}{2}\right]_0^1 \\
&= \frac{64}{3} \cdot -1 \cdot \frac{1}{2} = \boxed{-\frac{32}{3}}
\end{aligned}$$



Example (Similar to Exercise 5.249). Consider the region E bounded by the right circular cylinder with equation $r = 2\cos\theta$, the $r\theta$-plane, and the sphere $r^2+z^2 = 25$.

(a) Express the region E in cylindrical coordinates.

(b) Convert the integral $\iiint_E f(x,y,z)\,\mathrm{d}V$ to cylindrical coordinates.

Solution.

(a) Sketching the region E, we can determine that

$$E = \{(r,\theta,z) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta,\ 0 \le z \le \sqrt{25-r^2}\}$$

(b) The triple integral is

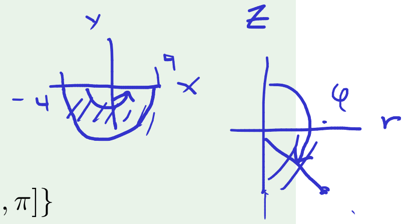
$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\int_0^{\sqrt{25-r^2}} f(r\cos\theta, r\sin\theta, z)\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

Example (Similar to Exercise 5.270)**.** Evaluate the triple integral $\iiint_B f(x,y,z)\,\mathrm{d}V$ over the solid B.

$f(x,y,z) = 2 - \sqrt{x^2+y^2+z^2}$ $= 2-\rho$

$B = \{(x,y,z) \mid x^2+y^2+z^2 \le 16,\ y \le 0,\ z \le 0\}$

$\rho^2 \le 16 \Rightarrow \rho \in [0,4]$



Solution. Converting the region to spherical, we get

$$B = \{(\rho,\theta,\varphi) \mid \rho \in [0,4],\ \theta \in [-\pi,0],\ \varphi \in [\frac{\pi}{2},\pi]\}$$

Thus

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_{-\pi}^{0}\int_{\frac{\pi}{2}}^{\pi}\int_0^4 (2-\rho)\underbrace{\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta}_{= dV}$$
$$= \int_0^4 (2\rho^2-\rho^3)\,\mathrm{d}\rho \cdot \int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,\mathrm{d}\varphi \cdot \int_{-\pi}^{0} 1\,\mathrm{d}\theta$$
$$= -\frac{64}{3}\cdot 1\cdot \pi = \boxed{-\frac{64\pi}{3}}$$

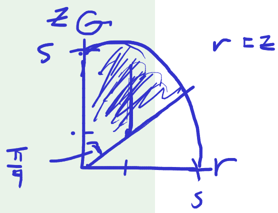
Example (Similar to Exercise 5.280)**.** Find the volume of the solid E whose boundaries are given in rectangular coordinates.

$$E = \{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le \sqrt{25-x^2-y^2},\ x \le 0,\ y \le 0\}$$

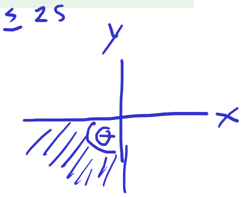
$0 \le r \le z$ $x^2+y^2+z^2 \le 25$

Solution. Converting to spherical, this region is $0 \le \rho \le 5$

$$E = \{(\rho,\varphi,\theta) \mid \rho \in [0,5], \varphi \in [0,\frac{\pi}{4}], \theta \in [\pi, \frac{3\pi}{2}]\}$$



$z^2+r^2 \le 25$



The compute the volume, we integrate:

$$\begin{aligned}
\iiint_E 1\,\mathrm{d}V &= \int_{\pi}^{3\pi/2}\int_0^{\pi/4}\int_0^5 \underbrace{\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta}_{= dV} \\
&= \int_0^5 \rho^2\,\mathrm{d}\rho \cdot \int_0^{\pi/4} \sin\varphi\,\mathrm{d}\varphi \cdot \int_{\pi}^{3\pi/2} 1\,\mathrm{d}\theta \\
&= \frac{125}{3}\cdot\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{\pi}{2} \\
&= \boxed{\frac{125\pi(2-\sqrt{2})}{12}}
\end{aligned}$$

1.6 Hints for Exercises

If you set up and compute the integrals correctly, you should get:

5.241: $\dfrac{9\pi}{8}$

5.270: $-\dfrac{45\pi}{4}$

5.280: $\dfrac{16\pi(2-\sqrt{2})}{3}$

Disclaimer For full credit, all work must be shown. Use these answers to double check your work. Just writing the answer down without any work should result in zero points for the problem.