Contents

1 §5.7: Changes of Variables in Multiple Integrals

1.1 Planar Transformations

A **planar transformation** T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of $\mathbb{R}^2$.

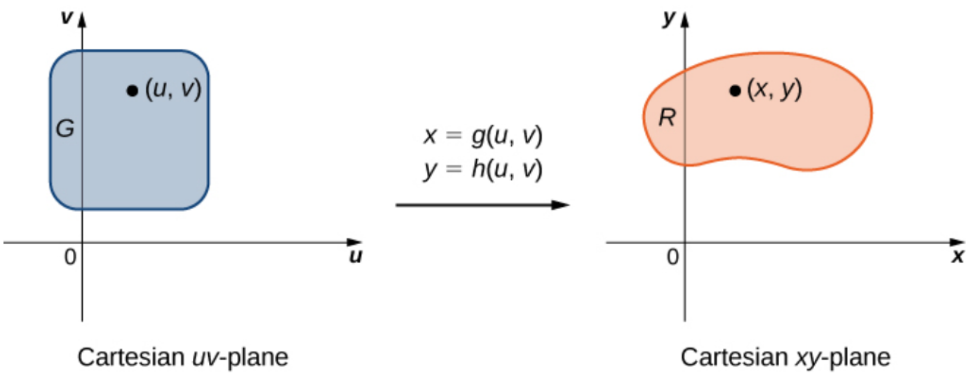
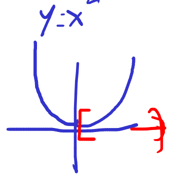


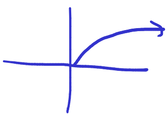
Figure 5.71 The transformation of a region G in the uv-plane into a region R in the xy-plane.

In the above figure, the region G in the uv-plane is transformed into the region R in the xy-plane by the change of variables $x = g(u, v)$ and $y = h(u, v)$. This is sometimes written $x = x(u, v)$ and $y = y(u, v)$. We will typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, h_v all exist and are continuous.

$y = x^2$



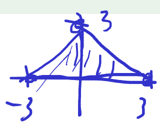
Definition. A transformation $T : G \to R$, defined as $T(u, v) = (x, y)$ is said to be a **one-to-one transformation** if no two points map to the same image point.



Example (Similar to Exercise 5.363)**.** Determine whether the transformation T : $S \to R$ ~~$S \to R$~~ is one-to-one or not.

$x = u^2, y = v^2$, where S is the triangle of vertices $(-3, 0), (3, 0), (0, 3)$.

$(u,v) \xrightarrow{T} (u^2, v^2)$



To show 1-1, must show $T(u_1,v_1) = T(u_2,v_2)$

$\Rightarrow (u_1,v_1) = (u_2,v_2)$

Solution. This is *not* one-to-one, since T maps both points $(-3,0)$ and $(3,0)$ to the point $(9,0)$.

Example. The transformation $T(r,\theta) = (x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$:

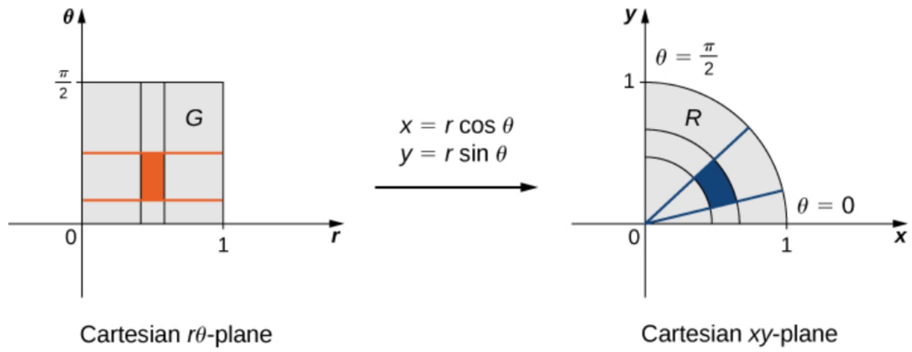


Figure 5.72 A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

1.2 Jacobian

As previously stated each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u, h_v exist and are continuous. A transformation that has this property is called a C^1 **transformation**.

Definition. The **Jacobian** of the C^1 transformation $T(u,v) = (\underbrace{g(u,v)}_{X}, \underbrace{h(u,v)}_{Y})$ is denoted by $J(u,v)$ and is defined by the 2×2 determinant

$$J(u,v) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Remark. The Jacobian is sometimes denoted as $\boxed{\dfrac{\partial(x,y)}{\partial(u,v)}}$ (without bars).

Remark. There is a general property of matrices/determinants that transposing the matrix (swapping rows and columns, aka flipping across main diagonal) does not change the determinant. Written mathematically, $\det(A^t) = \det(A)$. This means that the Jacobian can also be written as

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Remark. I will also use $\mathrm{Jac}(T)$ to denote the Jacobian of the transformation T.

Example (Checking the Jacobian for polar)**.** Find the Jacobian of the transformation $T(r,\theta) = (r\cos\theta, r\sin\theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$.

Solution.

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

Example (Similar to Exercise 5.381)**.** Find the Jacobian of the transformation $T(u,v) = (u^2 - v^2, uv)$, where $x = u^2 - v^2$ and $y = uv$.

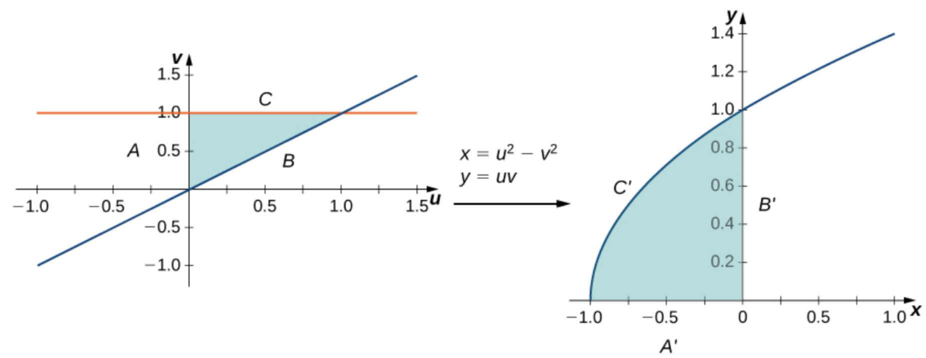


Figure 5.73 A triangular region in the uv-plane is transformed into an image in the xy-plane.

Solution.

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = \boxed{2u^2 + 2v^2}$$

1.3 Change of Variables for Double Integrals

Under the change of variables $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u,v)\Delta u \Delta v$$

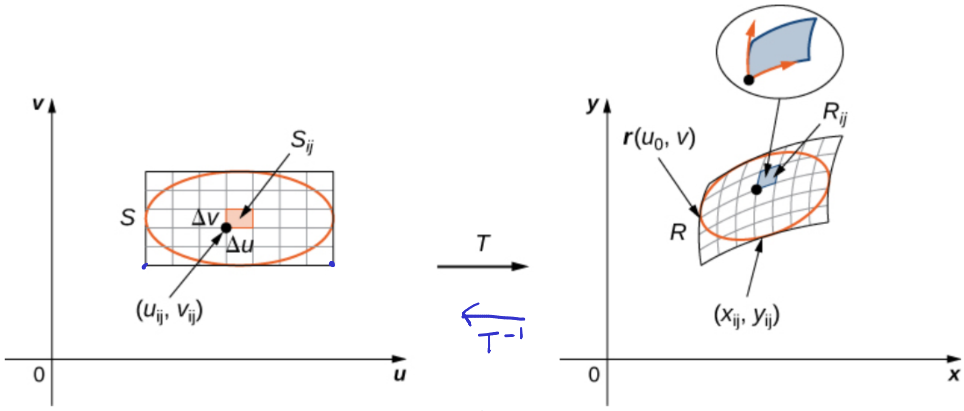


Figure 5.75 The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Theorem 1 (Change of Variables for Double Integrals)
Let $T(u, v) = (x, y)$ where $x = g(u, v)$ and $y = h(u, v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane; it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x, y)\,\mathrm{d}A = \iint_S f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v$$

With this theorem for double integrals, we can change the variables from (x, y) to (u, v) in a double integral simply by replacing

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v$$

when we use the substitutions $x = g(u, v)$ and $y = h(u, v)$ and then change the limits of integration accordingly.

Remark (Important remark on mapping direction)**.** Notice in the above, we have $T\colon (u, v) \to (x, y)$, where the region is easier to integrate in the uv-plane, harder to integrate in the xy-plane.

The book then seems to reverse the direction of the transformation in its examples, becoming $T^{-1}\colon (x, y) \to (u, v)$. This makes some sense, because such a mapping (going from the difficult region in the xy-plane to a simpler region in the uv-plane) can be easier to construct. But in order to utilize the theorem above, the transformation needs to be in the correct direction, so the book then does the process of solving for the inverse of T^{-1}, which is the transformation $T\colon (u, v) \to (x, y)$, and then compute the Jacobian of T so that the formula applies.

There is an easier way:

$$\mathrm{Jac}(T) = \mathrm{Jac}(T^{-1})^{-1} = \frac{1}{\mathrm{Jac}(T^{-1})}$$

Remark (On orientation reversal)**.** The Jacobian itself can be negative, which means that the transformation reverses the orientation. In the integral formulas, the absolute value of the Jacobian is typically used. This is fine, because two negatives arise in such a transformation. One comes from the Jacobian, one comes from the parametrization of the integration of the region, which will be reversed from the standard order.

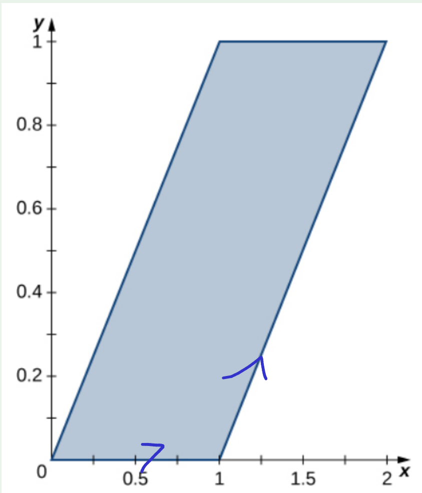
Example (Similar to Exercise 5.390)**.** Use the transformation $\underbrace{u = y - x,\ v = y}$ to evaluate the integral

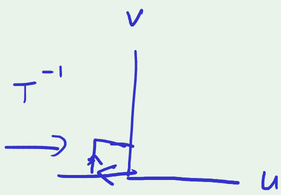
$$\iint_R y(y-x)^2 \, \mathrm{d}A$$

$=: T^{-1} : (x,y) \to (u,v)$

$R \to S$

on the parallelogram R of vertices $(0,0),(1,0),(2,1),(1,1)$ shown in the following figure.

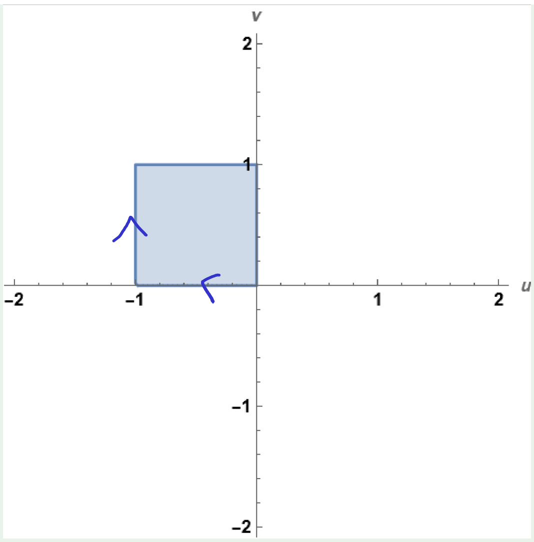




Solution. Note that the transformation given is going from $(x,y) \to (u,v)$, so it is the inverse transformation T^{-1}. The image of R under this map sends

$$\begin{aligned}
(0,0) &\mapsto (0,0) \\
(1,0) &\mapsto (-1,0) \\
(2,1) &\mapsto (-1,1) \\
(1,1) &\mapsto (0,1)
\end{aligned}$$

Visually, the region $T^{-1}(R) =: S$ is



- Computing the Jacobian the fast way:

$$\operatorname{Jac}(T^{-1}) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

  so $\operatorname{Jac}(T) = \frac{1}{\operatorname{Jac}(T^{-1})} = -1$.

- Computing the Jacobian the textbook way: We can solve for the inverse transformation

$$\overset{T^{-1}}{\begin{cases} u = y - x \\ v = y \end{cases}} \implies \overset{T}{\begin{cases} x = v - u \\ y = v \end{cases}} \qquad x = y - u$$

  so

$$\operatorname{Jac}(T) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

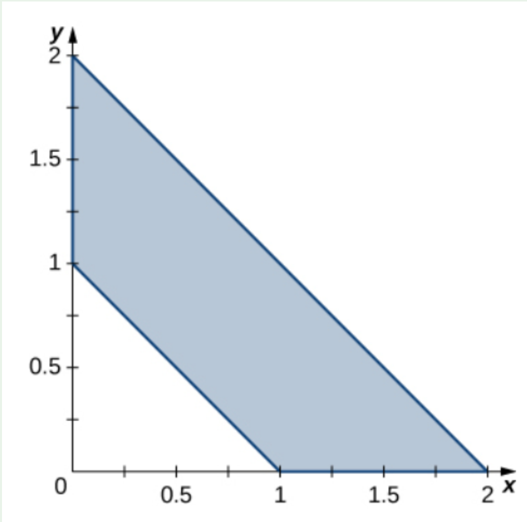
Either way, we got the same Jacobian. Computing the desired integral,

$$\begin{aligned} \iint_R y(y-x)^2 \, dA &= \iint_S vu^2 |\operatorname{Jac}(T)| \, dA \\ &= \int_0^1 \int_{-1}^0 vu^2 |-1| \, du \, dv \\ &= \int_{-1}^0 u^2 \, du \cdot \int_0^1 v \, dv \\ &= \frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{6}} \end{aligned}$$

Example (Similar to Exercise 5.396)**.** Use the transformation $\underbrace{u = x + y,\ v = x - y}_{T^{-1}: (x,y) \to (u,v)}$ to evaluate the integral

$$\iint_R (x-y)^4 e^{x+y}\, \mathrm{d}A$$

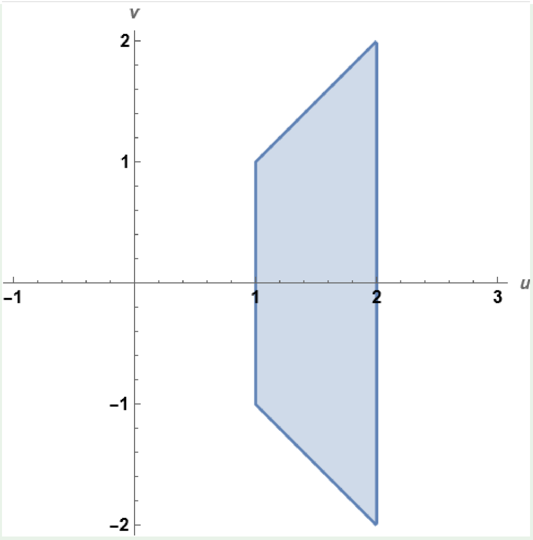
on the trapezoidal region R determined by the points $(1,0), (2,0), (0,2), (0,1)$. Shown in the following figure.



Solution. Note that the transformation given is going from $(x,y) \to (u,v)$, so it is the inverse transformation T^{-1}. The image of R under this map sends

$$\begin{aligned}(1,0) &\mapsto (1,1)\\ (2,0) &\mapsto (2,2)\\ (0,1) &\mapsto (1,-1)\\ (0,2) &\mapsto (2,-2)\end{aligned}$$

Visually, the region $T^{-1}(R) =: S$ is



$$\int_1^2 \int_{-u}^{u} f \, |Jac(T)| \, dv\, du$$

- Computing the Jacobian the fast way:

$$\mathrm{Jac}(T^{-1}) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so $\mathrm{Jac}(T) = \frac{1}{\mathrm{Jac}(T^{-1})} = -\frac{1}{2}$.

- Computing the Jacobian the textbook way: We can solve for the inverse transformation

$$\overbrace{\begin{cases} u = x + y \\ v = x - y \end{cases}}^{T^{-1}} \implies \overbrace{\begin{cases} x = \dfrac{1}{2}(u+v) \\ y = \dfrac{1}{2}(u-v) \end{cases}}^{T}$$

so

$$\mathrm{Jac}(T) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Either way, we got the same Jacobian. Computing the desired integral,

$$\begin{aligned}
\iint_R (x-y)^4 e^{x+y}\,\mathrm{d}A &= \iint_S v^4 e^u |\operatorname{Jac}(T)|\,\mathrm{d}A \\
&= \int_1^2 \int_{-u}^{u} v^4 e^u \left|-\frac{1}{2}\right| \mathrm{d}v\,\mathrm{d}u \\
&= \frac{1}{2}\int_1^2 e^u \left[\frac{v^5}{5}\right]_{v=-u}^{u} \mathrm{d}u \\
&= \frac{1}{5}\int_1^2 u^5 e^u\,\mathrm{d}u \\
&= \frac{1}{5}\, e^u(u^5 - 5u^4 + 20u^3 - 60u^2 + 120u - 120)\Big|_1^2 \\
&= \boxed{\frac{4}{5}e(11-2e)}
\end{aligned}$$

	D	I
+	u^5	e^u
−	$5u^4$	e^u
+	$20u^3$	e^u
−	$60u^2$	e^u
+	$120u$	e^u
−	120	e^u
+	0	e^u

1.4 Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space by a one-to-one C^1 transformation $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$. Then any function $F(x,y,z)$ defined on D can be thought of as another function $H(u,v,w)$ that is defined on G :

$$F(x,y,z) = F(g(u,v,w), h(u,v,w), k(u,v,w)) = H(u,v,w).$$

Now we need to define the Jacobian for three variables.

Definition. The Jacobian determinant $J(u,v,w)$ in three variables is defined as follows:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

With this, we can establish the theorem that describes change of variables for triple integrals.

Theorem 2 (Change of Variables for Triple Integrals)
Let $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-plane into the region D in the xyz-plane. As in the two-dimensional case, if F is continuous on D, then

$$\begin{aligned}\iiint_R F(x,y,z)\,\mathrm{d}V &= \iiint_G F\big(g(u,v,w), h(u,v,w), k(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\ &= \iiint_G H(u,v,w)|J(u,v,w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\end{aligned}$$

Example (Formulas for triple integrals in Cylindrical and Spherical)**.** We derive the triple integral formulas for

(a) cylindrical and

(b) spherical coordinates

Solution.

(a) For cylindrical, the transformation is $T(r,\theta,z) = (x,y,z)$ from the Cartesian $r\theta z$-plane to the Cartesian xyz-plane. Here $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The Jacobian for the transformation is

$$\begin{aligned}J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r.\end{aligned}$$

Thus we get

$$\iiint_D f(x,y,z)\,\mathrm{d}V = \iiint_G f(r\cos\theta, r\sin\theta, z)\; r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

(b) For spherical coordinates, the transformation is $T(\rho,\theta,\varphi) = (x, y, z)$ from the Cartesian $\rho\theta\varphi$-plane to the Cartesian xyz-plane. Here, $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$. The Jacobian for the transformation is

$$\begin{aligned}
J(\rho,\theta,\varphi) = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \varphi} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \varphi} \end{vmatrix} \\
&= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & +\rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\cancel{\theta}\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\
&= \cos\varphi \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \end{vmatrix} \\
&\qquad - \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix} \\
&= \cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right) \\
&\qquad - \rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right) \\
&= -\rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) - \rho^2\sin\varphi\sin^2\varphi\left(\sin^2\theta + \cos^2\theta\right) \\
&= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi \\
&= -\rho^2\sin\varphi\left(\cos^2\varphi + \sin^2\varphi\right) = -\rho^2\sin\varphi.
\end{aligned}$$

To fix orientation issues, we take the absolute value of this, so $|J(\rho,\theta,\varphi)| = |-\rho^2\sin\varphi| = \rho^2\sin\varphi$. Note that this could have been fixed in the beginning if we used the order $\rho\varphi\theta$ instead of $\rho\theta\varphi$. Thus we get

$$\iiint_D f(x,y,z)\,\mathrm{d}V = \iiint_G f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta.$$

1.5 Hw Hints

5.390 $-\dfrac{1}{2}$

5.396 $\dfrac{2}{3}e(e+1)$. Tabular integration by parts will help. Otherwise, you'll need to do integration by parts 3 times. (Search for "bprp DI method" on YouTube if you are not familiar with the tabular approach to integration by parts.)