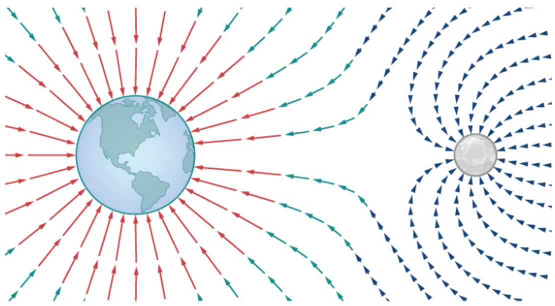
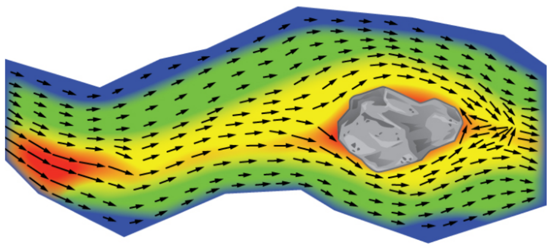
Contents

1 §6.1: Vector Fields

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents.

Some examples of vector fields:





Definition. A **vector field F** assigns a vector to every point in space.

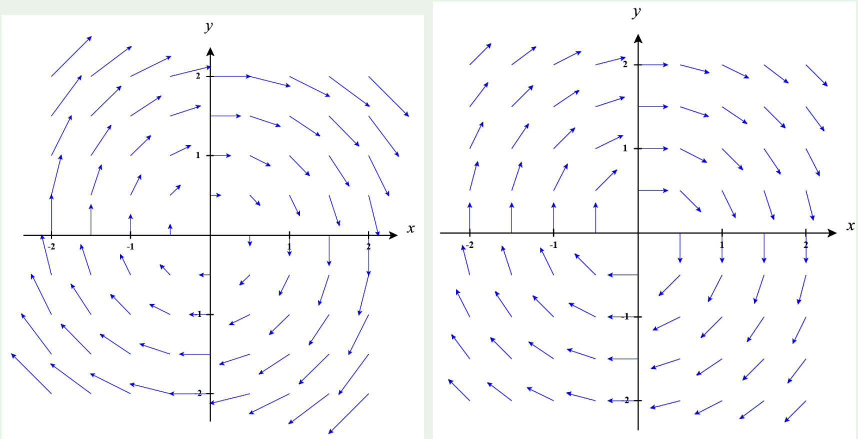
- A vector field in $\mathbb{R}^2$ assigns to every point in a subset of $\mathbb{R}^2$ a two-dimensional vector $\mathbf{F}(x, y)$.
- A vector field in $\mathbb{R}^3$ assigns to every point in a subset of $\mathbb{R}^3$ a three-dimensional vector $\mathbf{F}(x, y, z)$.

A **unit vector field** is a vector field where each vector in the field is of unit length.

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field, then the corresponding *unit* vector field is $\left\langle \frac{P}{\|\mathbf{F}\|}, \frac{Q}{\|\mathbf{F}\|}, \frac{R}{\|\mathbf{F}\|} \right\rangle$. This process of dividing $\mathbf{F}$ by its magnitude to form the unit vector field $\frac{\mathbf{F}}{\|\mathbf{F}\|}$ is called **normalizing** the vector field $\mathbf{F}$.

Example (Example of a unit vector field)**.**

$$\vec{G}(x,y) = \langle y, -x \rangle, \quad \mathbf{F}(x,y) = \left\langle \frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}} \right\rangle$$



1.1 Software to graph vector fields

A quick search should allow you to find ways to graph vector fields (both 2D and 3D) using Desmos, GeoGebra, or WolframAlpha. CalcPlot3D also has such capabilities.

1.2 Gradient Fields

Definition. A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a **gradient field** (also called a **conservative field**) if there exists a scalar function f such that $\nabla f = \mathbf{F}$. In this situation, f is called a **potential function** for $\mathbf{F}$.

Such vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved (e.g., gravitational fields and electric fields associated with a static charge).

More will be said about conservative vector fields in §6.3.

Remark. In some applications, a potential function f for $\mathbf{F}$ is defined instead as a function such that $-\nabla f = \mathbf{F}$. This is the case for certain contexts in physics, for example.

A natural question to ask is whether a conservative vector field may have more than one potential function, and if so, how are they related? The following theorem answers this question.

Theorem 1 (6.1: Uniqueness of Potential Functions)

Let $\mathbf{F}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\nabla f = \nabla g = \mathbf{F}$. Then there is a constant C such that $f = g + C$.

Conservative vector fields satisfy the **cross-partial property**.

Theorem 2 (6.2: The Cross-Partial Property of Conservative Vector Fields)

Let $\mathbf{F}$ be a vector field in two or three dimensions such that the component functions of $\mathbf{F}$ have continuous second-order mixed-partial derivatives on the domain of $\mathbf{F}$.

- If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

x y z

- If $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof. Since $\mathbf{F}$ is conservative, there is a function $f(x, y)$ such that $\nabla f = \mathbf{F}$. Therefore, by definition of the gradient, $f_x = P$ and $f_y = Q$. By Clairaut's theorem,

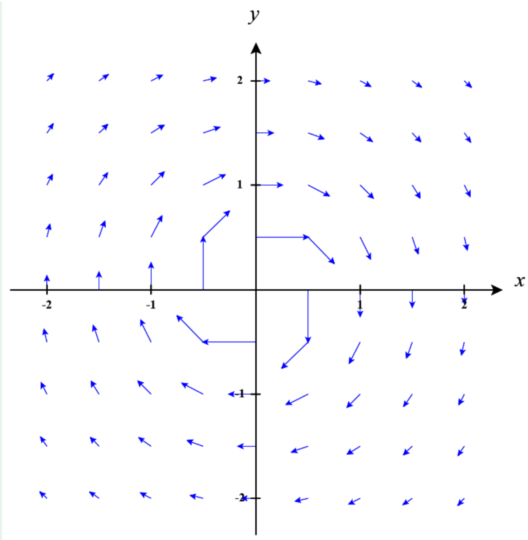
$$P_y = f_{xy} = f_{yx} = Q_x$$

The proof for conservative vector fields in $\mathbb{R}^3$ is similar. □

Thus if $\mathbf{F}$ is conservative, then $\mathbf{F}$ has the cross-partial property.
Warning: The converse does not hold. An additional assumption is needed.

Example (Standard example of a vector field that satisfies the cross partial property but is not conservative).

$$\mathbf{F}(x, y) = \left\langle \frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right\rangle$$



Checking it satisfies the cross-partial property:

$$\frac{\partial}{\partial y}\frac{y}{x^2+y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$
$$\frac{\partial}{\partial x}\frac{-x}{x^2+y^2} = \frac{(x^2+y^2)(-1) - (-x)(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

In §6.3, we will revisit this vector field and show that it is not conservative (an additional property of conservative vector fields is needed).