Contents

1 §6.2: Line Integrals

In this section, we introduce two types of integrals over curves: integrals of functions and integrals of vector fields. These are traditionally called **line integrals**, although it would be more appropriate to call them “curve” or “path” integrals.

1.1 Scalar Line Integrals

The **scalar line integral** $\int_C f(x, y, z)\,\mathrm{d}s$ over a curve C is defined as a Riemann sum.

Subdivide the curve C into N consecutive arcs $C_1, \ldots, C_n$, and choose a sample point P_i on each arc C_i.

Partition of C into N small arcs   Choice of sample points P_i in each arc

These form the Riemann sum

$$\sum_{i=1}^{N} f(P_i)\,\mathrm{length}(C_i) = \sum_{i=1}^{N} f(P_i)\Delta s_i$$

where Δs_i is the length of C_i.

The line integral of f over C is then the limit of these Riemann sums as the maximum of the lengths Δs_i approaches zero:

$$\int_C f(x, y, z)\,\mathrm{d}s = \lim_{\{\Delta s_i\}\to 0} \sum_{i=1}^{N} f(P_i)\Delta s_i$$

This definition also applies to functions $f(x, y)$ of two variables.

The scalar line integral of the function $f(x, y, z) = 1$ is simply the length of C:

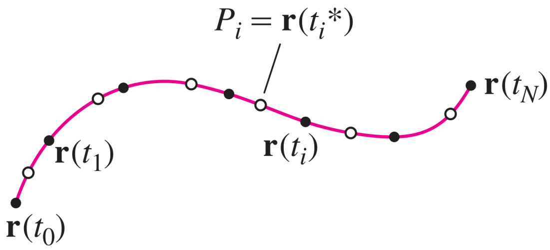
$$\boxed{\int_C 1 \, \mathrm{d}s = \text{length}(C)}$$

In practice, line integrals are computed using parametrizations. Suppose C has a parametrization $\mathbf{r}(t)$ for $a \le t \le b$ with continuous derivative $\mathbf{r}'(t)$.

We divide C into N consecutive arcs $C_1, \ldots, C_N$ corresponding to a partition of the interval $[a, b]$:

$$a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$$

so that C_i is parametrized by $\mathbf{r}(t)$ for $t_{i-1} \le t \le t_i$, and choose sample points $P_i = \mathbf{r}(t_i^*)$ with t_i^* in $[t_{i-1}, t_i]$.



According to the arc length formula,

$$\text{length}(C_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} \left\|\mathbf{r}'(t)\right\| \mathrm{d}t$$

Because $\mathbf{r}'(t)$ is continuous, the function $\left\|\mathbf{r}'(t)\right\|$ is nearly constant on $[t_{i-1}, t_i]$ if the length $\delta t_i = t_i - t_{i-1}$ is small, and thus, $\displaystyle\int_{t_{i-1}}^{t_i} \left\|\mathbf{r}'(t)\right\| \mathrm{d}t \approx \left\|r'(t_i^*)\right\| \Delta t_i$. This gives us the approximation

$$\sum_{i=1}^{N} f(P_i)\Delta s_i \approx \sum_{i=1}^{N} f(\mathbf{r}(t_i^*))\left\|\mathbf{r}'(t_i^*)\right\|\Delta t_I$$

The sum on the right is a Riemann sum that converges to the integral

$$\int_a^b f(\mathbf{r}(t))\left\|\mathbf{r}'(t)\right\| \mathrm{d}t$$

as the maximum of the lengths Δt_i tends to zero. This gives us the following formula for the scalar line integral:

Theorem 1 (Computing a Scalar Line Integral)
Let $\mathbf{r}(t)$ be a parametrization of a curve C for $a \le t \le b$. If $f(x, y, z)$ and $\mathbf{r}'(t)$ are

continuous, then

$$\boxed{\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\left\|\mathbf{r}'(t)\right\|\mathrm{d}t}$$

The symbol $\mathrm{d}s$ is intended to suggest arc length s and is often referred to as the **line element** or **arc length differential**. In terms of a parametrization, we have the symbolic equation

$$\mathrm{d}s = \left\|\mathbf{r}'(t)\right\|\mathrm{d}t$$

where

$$\left\|\mathbf{r}'(t)\right\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Example (Scalar line integral)**.** Find $\int_C (x+y+z)\,\mathrm{d}s$, where C is the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le \pi$.

Solution.

$$\begin{aligned} \mathbf{r}(t) &= \langle \cos t, \sin t, t\rangle \qquad t \in [0,\pi] \\ \mathbf{r}'(t) &= \langle -\sin t, \cos t, 1\rangle \\ \left\|\mathbf{r}'(t)\right\| &= \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2} \end{aligned}$$

and

$$\begin{aligned} f(x,y,z) &= x+y+z \\ f(\mathbf{r}(t)) &= f(\cos t, \sin t, t) = \cos t + \sin t + t \end{aligned}$$

so

$$\begin{aligned} \int_C f(x,y,z)\,\mathrm{d}s &= \int_0^\pi f(\mathbf{r}(t))\left\|\mathbf{r}'(t)\right\|\mathrm{d}t \\ &= \int_0^\pi (\cos t + \sin t + t)\sqrt{2}\,\mathrm{d}t \\ &= \sqrt{2}\left(\sin t - \cos t + \frac{1}{2}t^2\right)\Big|_0^\pi \\ &= \boxed{2\sqrt{2} + \frac{\sqrt{2}}{2}\pi^2} \end{aligned}$$

1.2 Applications of the Scalar Line Integral

The general principle is that the integral of density is the total quantity.

If we view the curve C as a wire, and $\rho(x, y, z)$ as the mass density of the wire, then the total mass of the wire is given by the integral

$$\int_C \rho(x, y, z)\,\mathrm{d}s$$

If ρ instead stood for the charge density, then this integral would measure total charge.

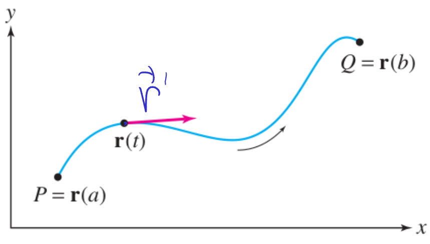
1.3 Vector Line Integrals

Work is an example of a quantity represented by a vector line integral.

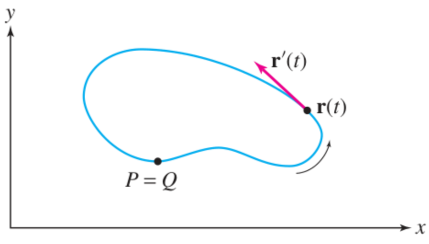
An important difference between vector and scalar line integrals is that vector line integrals depend on the direction along the curve.

A specified direction along a curve C is called an **orientation**. This direction is called the **positive** direction along C, the opposite direction is the **negative** direction, and C itself is called an **oriented curve**.

A curve is **closed** if there is a parametrization $\mathbf{r}(t)$, $a \le t \le b$, such that $\mathbf{r}(a) = \mathbf{r}(b)$, and the curve is traversed exactly once. In other words, closed curves are loops.



(A) Oriented path from P to Q

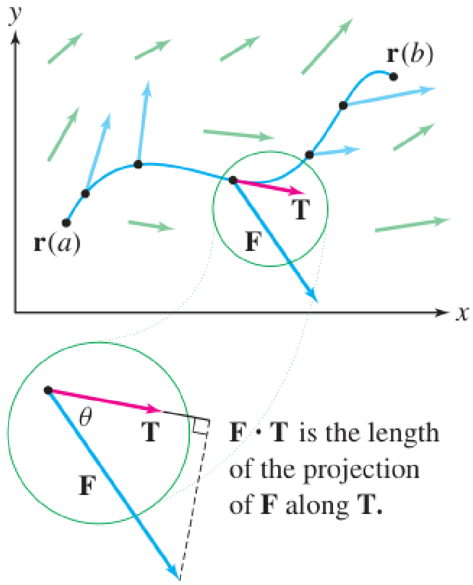


(B) A closed oriented path

The line integral of a vector field $\mathbf{F}$ over a curve C is defined as the scalar line integral of the tangential component of $\mathbf{F}$.

Definition (Vector Line Integral)**.** The line integral of a vector field $\mathbf{F}$ along an oriented curve C is the integral of the tangential component of $\mathbf{F}$:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\mathbf{F} \cdot \mathbf{T})\,\mathrm{d}s$$



We use parametrizations to evaluate vector line integrals, but there is one important difference with the scalar case: The parametrization $\mathbf{r}(t)$ must be *positively oriented*; that is, $\mathbf{r}(t)$ must trace C in the positive direction.

Recall that

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

Then we have

$$(\mathbf{F} \cdot \mathbf{T})\, \mathrm{d}s = \left(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \right) \|\mathbf{r}'(t)\|\, \mathrm{d}t = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t$$

Thus we have

Theorem 2 (Computing a Vector Line Integral)
If $\mathbf{r}(t)$ is a parametrization of an oriented curve C for $a \le t \le b$, then

$$\boxed{\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t}$$

It is useful to think of $\mathrm{d}\mathbf{r}$ as a "vector line element" or "vector differential" that is related to the parametrization by the symbolic equation

$$\boxed{\mathrm{d}\mathbf{r} = \mathbf{r}'(t)\, \mathrm{d}t = \langle x'(t), y'(t), z'(t) \rangle\, \mathrm{d}t}$$

Example (Similar to Exercise 6.74)**.** Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F} = \langle z, y^2, x \rangle$ and C is parametrized (in the positive direction) by $\mathbf{r}(t) = \langle t+1, e^t, t^2 \rangle$ for $0 \le t \le 2$.

Solution.

$$\mathbf{r}(t) = \left\langle t+1, e^t, t^2 \right\rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \left\langle t^2, e^{2t}, t+1 \right\rangle$$
$$\mathbf{r}'(t) = \left\langle 1, e^t, 2t \right\rangle$$

so

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^2 \left\langle t^2, e^{2t}, t+1 \right\rangle \cdot \left\langle 1, e^t, 2t \right\rangle \mathrm{d}t$$
$$= \int_0^2 (e^{3t} + 3t^2 + 2t)\,\mathrm{d}t$$
$$= \frac{1}{3}e^{3t} + t^3 + t^2 \Big|_0^2 = \frac{1}{3}e^6 + 8 + 4 - \frac{1}{3}$$
$$= \frac{1}{3}(e^6 - 1) + 12$$

Another standard notation for the vector line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is

$$\boxed{\int_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z}$$

In this notation, we write $\mathrm{d}\mathbf{r}$ as a vector differential

$$\mathrm{d}\mathbf{r} = \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z \rangle$$

so that

$$\mathbf{F} \cdot \mathrm{d}\mathbf{r} = \langle F_1, F_2, F_3 \rangle \cdot \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z \rangle = F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z$$

In terms of a parametrization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$,

$$\mathrm{d}\mathbf{r} = \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle \mathrm{d}t$$
$$\mathbf{F} \cdot \mathrm{d}\mathbf{r} = \left(F_1(\mathbf{r}(t))\frac{\mathrm{d}x}{\mathrm{d}t} + F_2(\mathbf{r}(t))\frac{\mathrm{d}y}{\mathrm{d}t} + F_3(\mathbf{r}(t))\frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t$$

So we have the following formula:

$$\boxed{\int_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z = \int_a^b \left(F_1(\mathbf{r}(t))\frac{\mathrm{d}x}{\mathrm{d}t} + F_2(\mathbf{r}(t))\frac{\mathrm{d}y}{\mathrm{d}t} + F_3(\mathbf{r}(t))\frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t}$$

Example (Similar to Exercise 6.57). Let C be the ellipse parametrized by $\mathbf{r}(\theta) = \langle 5 + 4\cos\theta, 3 + 2\sin\theta \rangle$ for $0 \le \theta \le 2\pi$. Calculate

$$\int_C 2y \, \mathrm{d}x - 3 \, \mathrm{d}y$$

$F = \langle 2y, -3 \rangle$

Solution.

$$\begin{aligned} \mathbf{r}(\theta) &= \langle \overset{x}{5 + 4\cos\theta}, \overset{y}{3 + 2\sin\theta} \rangle \\ \mathbf{r}'(\theta) &+ \langle \underset{dx}{-4\sin\theta}, \underset{dy}{2\cos\theta} \rangle \end{aligned}$$

so

$$\begin{aligned} \int_C 2y \, \mathrm{d}x - 3 \, \mathrm{d}y &= \int_0^{2\pi} \big(2(3 + 2\sin\theta)(-4\sin\theta) - 3(2\cos\theta) \big) \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \big(24\sin\theta + 16\sin^2\theta + 6\cos\theta \big) \, \mathrm{d}\theta \\ &= -16 \int_0^{2\pi} \sin^2\theta \, \mathrm{d}\theta \\ &= \boxed{-16\pi} \end{aligned}$$

Theorem 3 (Properties of Vector Line Integrals)
Let C be a smooth oriented curve, and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields.

(i) Linearity:

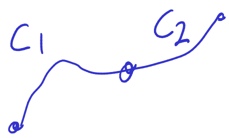
$$\int_C (\mathbf{F} + \mathbf{G}) \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r}$$

$$\int_C k\mathbf{F} \cdot \mathrm{d}\mathbf{r} = k \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \qquad (k \text{ a constant})$$

(ii) Reversing orientation: $\displaystyle\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $-C$ denotes the curve with opposite orientation.

(iii) Additivity: If C is the union of n smooth curves $C_1 + \cdots + C_n$, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$



1.4 Applications of Vector Line Integrals

1.4.1 Work

When force acts on an object moving along a curve, it makes sense to define the work W performed as the line integral

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

This is the work "performed by the field $\mathbf{F}$".

Example (Similar to Exercise 6.65)**.** Calculate the work performed by $\mathbf{F}$ in moving a particle from $(0,0,0)$ to $(4,8,1)$ along the path $\mathbf{r}(t) = \langle t^2, t^3, t\rangle$ for $1 \le t \le 2$ in the presence of a force field $\mathbf{F} = \langle x^2, -z, -yz^{-1}\rangle$.

Solution.

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) &= \mathbf{F}(t^2, t^3, t) = \left\langle t^4, -t, -t^2 \right\rangle \\
\mathbf{r}'(t) &= \left\langle 2t, 3t^2, 1 \right\rangle \\
\mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \left\langle t^4, -t, -t^2 \right\rangle \cdot \left\langle 2t, 3t^2, 1 \right\rangle \mathrm{d}t = (2t^5 - 3t^3 - t^2)\,\mathrm{d}t
\end{aligned}$$

The work performed by the force field is thus

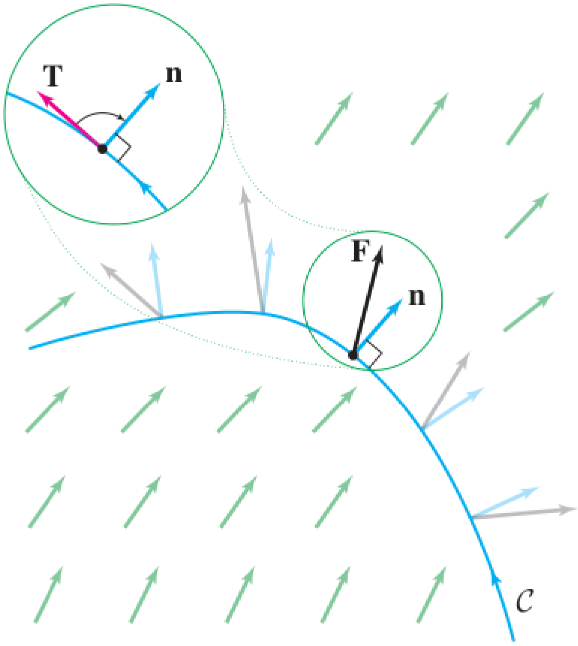
$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^2 (2t^5 - 3t^3 - t^2)\,\mathrm{d}t = \frac{-89}{12}$$

1.4.2 Flux

Line integrals are also used to compute the **flux across a plane curve**, defined as the *normal component* of the vector field, rather than the tangential component.

$$\int_C \vec{F} \cdot \vec{T}\, ds$$

$$\int_C \vec{F} \cdot \vec{N}\, ds$$



Suppose that a plane curve C is parametrized by $\mathbf{r}(t)$ for $a \le t \le b$, and let

$$\mathbf{N} = \mathbf{N}(t) = \langle y'(t), -x'(t) \rangle, \qquad \mathbf{n}(t) = \frac{\mathbf{N}(t)}{\|\mathbf{N}(t)\|} = \frac{\mathbf{N}(t)}{\|\mathbf{r}'(t)\|}$$

These vectors are normal to C since the dot product of $\mathbf{N}$ with the tangent vector $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ is zero. The flux across C is the integral of the normal conponent $\mathbf{F} \cdot \mathbf{n}$, which can be obtained by integrating $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t)$ with respect to t:

$$\text{Flux across } C = \int_C (\mathbf{F} \cdot \mathbf{n})\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{N}(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t)\, dt$$

If $\mathbf{F}$ is the velocity field of a fluid (modeled as a two-dimensional fluid), then the flux is the quantity of fluid flowing across the curve per unit time.

Example (Similar to Exercise 6.90: Flux Across a Curve)**.** Calculate the flux of the velocity vector field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0 \rangle$ across the quarter ellipse $\mathbf{r}(t) = \langle 3\cos t, 6\sin t \rangle$ for $0 \le t \le \frac{\pi}{2}$.

Solution.

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -3\sin t, 6\cos t\rangle \\
\mathbf{N}(t) &= \langle 6\cos t, 3\sin t\rangle \\
\mathbf{v}(\mathbf{r}(t)) &= \left\langle 3 + 2(6\sin t) - (6\sin t)^2/3, 0\right\rangle \\
&= \left\langle 3 + 12\sin t - 12\sin^2 t, 0\right\rangle \\
\mathbf{v}(\mathbf{r}(t))\cdot\mathbf{N}(t) &= \left\langle 3 + 12\sin t - 12\sin^2 t, 0\right\rangle \cdot \langle 6\cos t, 3\sin t\rangle \\
&= 18\cos t + 72\sin t\cos t - 72\sin^2 t\cos t
\end{aligned}$$

The flux is thus

$$\int_a^b \mathbf{v}(\mathbf{r}(t))\cdot\mathbf{N}(t)\,\mathrm{d}t = \int_0^{\pi/2}(18\cos t + 72\sin t\cos t - 72\sin^2 t\cos t)\,\mathrm{d}t \quad = 18 + 36 - 24 = \boxed{30}$$