Contents

1 §6.3: Conservative Vector Fields

$\nabla f = \vec{F}$

In this section, we study conservative vector fields in more depth. When a curve $\mathcal{C}$ is closed, we often refer to the line integral as the **circulation** of **F** around $\mathcal{C}$ and denote it with the symbol $\oint$:

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

Figure 1: Path independence: If **F** is conservative, then the line integrals over $\mathbf{r}_1$ and $\mathbf{r}_2$ are equal

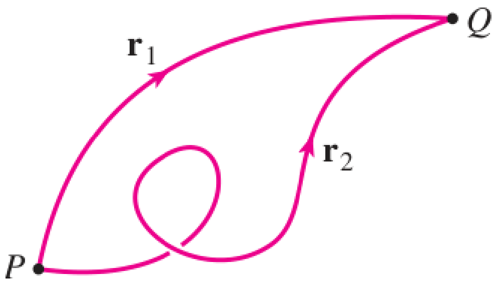
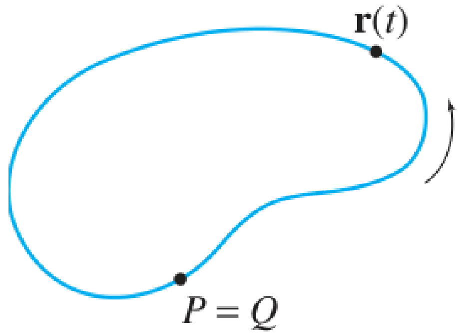


Figure 2: The circulation around a closed path is denoted $\oint_C \mathbf{F} \cdot d\mathbf{r}$



Theorem 1 (Fundamental Theorem for Line Integrals (FTLI))
Assume that $\mathbf{F} = \nabla f$ on a domain $\mathcal{D}$.

$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

(a) If **r** is a path along a curve $\mathcal{C}$ from P to Q in $\mathcal{D}$, then

$$\boxed{\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P)} \quad (1)$$

$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$

In particular, $\vec{F}$ is path independent.

(b) The circulation around a closed curve $\mathcal{C}$ (i.e., $P = Q$) is zero:

$$\boxed{\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0}$$

$f(Q) - f(P) = 0$

Proof. Let $\mathbf{r}(t)$ be a path along the curve $\mathcal{C}$ in $\mathcal{D}$ for $a \le t \le b$ with $\mathbf{r}(a) = P$ and $\mathbf{r}(b) = Q$. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = \int_a^b \boldsymbol{\nabla} f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t$$

However, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t)) = \boldsymbol{\nabla} f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

Thus, we can apply the Fundamental Theorem of Calculus:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\,\mathrm{d}t = f(\mathbf{r}(t))\Big|_a^b = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(Q) - f(P)$$

This proves (1). It also proves path independence, because the quantity $f(Q) - f(P)$ depends on the endpoints but not on the path $\mathbf{r}$. If $\mathbf{r}$ is a closed path, then $P = Q$ and $f(Q) - f(P) = 0$. □

Example. Let $\mathbf{F}(x, y, z) = \langle 2xy + z, x^2, x \rangle$.

(a) Verify that $f(x, y, z) = x^2y + xz$ is a potential function.

(b) Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathcal{C}$ is a curve from $P = (1, -1, 2)$ to $Q = (2, 2, 3)$.

Solution.

(a) The partial derivatives of $f(x, y, z) = x^2y + xz$ are the components of $\mathbf{F}$:

$$\frac{\partial f}{\partial x} = 2xy + z, \quad \frac{\partial f}{\partial y} = x^2, \quad \frac{\partial f}{\partial z} = x$$

Therefore, $\nabla f = \langle 2xy + z, x^2, x \rangle = \mathbf{F}$.

(b)

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= f(Q) - f(P) \\ &= f(2, 2, 3) - f(1, -1, 2) \\ &= \Big(2^2(2) + 2(3)\Big) - \Big(1^2(-1) + 1(2)\Big) = 13\end{aligned}$$

Example. Find a potential function for $\mathbf{F} = \langle 2x + y, x \rangle$ and use it to evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{r}$ is any path from $(1, 2)$ to $(5, 7)$.

Solution. We can see that $f(x, y) = x^2 + xy$ satisfies $\nabla f = \mathbf{F}$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + xy\right) = 2x + y, \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + xy\right) = x$$

Therefore, for any path $\mathbf{r}$ from $(1, 2)$ to $(5, 7)$,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(5, 7) - f(1, 2) = \left(5^2 + 5(7)\right) - \left(1^2 + 1(2)\right) = 57$$

P=Q

Example (Integral Around a Closed Path)**.** Let $f(x, y, z) = xy \sin(yz)$. Evaluate $\oint_{\mathcal{C}} \nabla f \cdot d\mathbf{r}$, where $\mathcal{C}$ is the closed curve in Figure 5.

Solution. By FTLI, the integral of a gradient vector around any closed path is zero. In other words, $\oint_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = 0$.

You might wonder whether there exist any path-independent vector fielsd other than the conservative ones. The answer is no. By the next theorem, a path-independent vector field is necessarily conservative.

Theorem 2

A vector field $\mathbf{F}$ on an open connected domain $\mathcal{D}$ is path-independent if and only if it is conservative.

Proof. The $\Longleftarrow$ direction is clear. For the $\Longrightarrow$ direction, see the proof of OpenStax Calc 3 Theorem 6.9. ☐

1.1 Finding Potential Functions

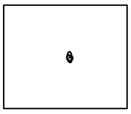
x, y, z

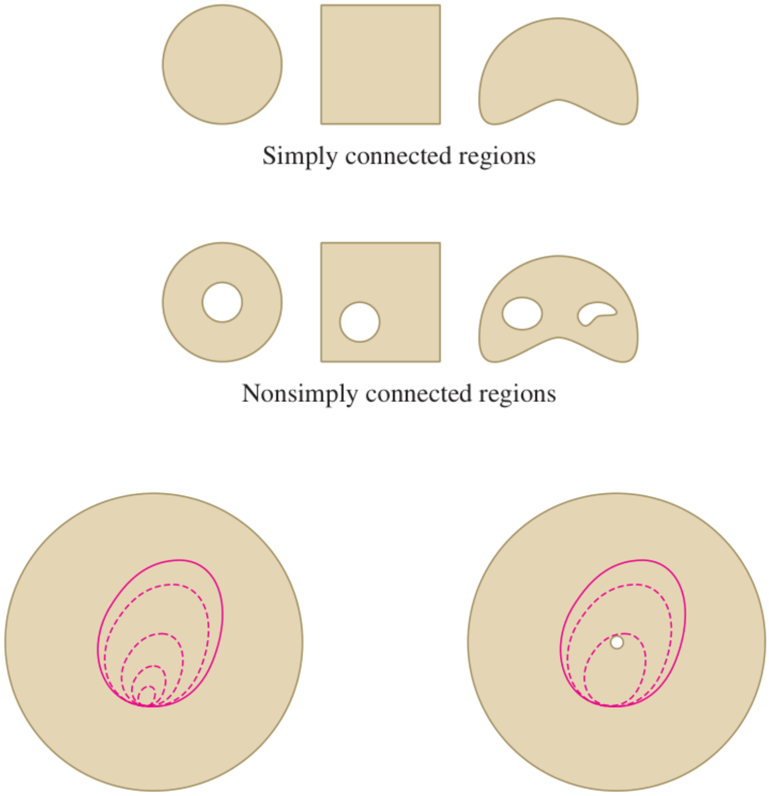
Recall that every conservative vector field $\mathbf{F} = \langle P, Q, R \rangle$ satisfies the cross-partial property:

$$P_y = Q_x \qquad P_z = R_x \qquad Q_z = R_y$$

This condition guarantees that $\mathbf{F}$ is conservative on domains $\mathcal{D}$ that are **simply connected**. Roughy speaking, this means that the domain does not have any "holes". More precisely, $\mathcal{D}$ is simply connected if every loop in $\mathcal{D}$ can be contracted to a point *while staying within* $\mathcal{D}$.

Figure 3: Simply connected means "no holes"





(A) Simply connected region: Any loop can be drawn down to a point within the region.

(B) Nonsimply connected region: a loop around the missing hole cannot be drawn tight without passing through the hole.

Theorem 3 (Existence of a Potential Function)
Let $\mathbf{F}$ be a vector field on a simply connected domain $\mathcal{D}$. If $\mathbf{F}$ satisfies the cross-partials property, then $\mathbf{F}$ is conservative.

Example (Finding a Potential Function). Show that

$$\mathbf{F} = \left\langle \overset{F_1}{2xy + y^3}, \overset{F_2}{x^2 + 3xy^2 + 2y} \right\rangle$$

is conservative and find a potential function.

Solution. First we observe that the cross-partial derivatives are equal:

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left(2xy + y^3\right) &&= 2x + 3y^2 \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + 3xy^2 + 2y\right) &&= 2x + 3y^2 \end{aligned}$$

Furthermore, $\mathbf{F}$ is defined on all of $\mathbb{R}^2$, which is a simply connected domain. Therefore, a potential function exists. Now, the potential function f satisfies

$$\frac{\partial f}{\partial x} = F_1(x,y) = 2xy + y^3$$

This tells us that f is an antiderivative of $F_1(x,y)$, regarded as a function of x alone:

$$\begin{aligned} f(x,y) &= \int F_1(x,y)\,\mathrm{d}x \\ &= \int \left(2xy + y^3\right)\mathrm{d}x \\ &= x^2y + xy^3 + \underline{g(y)} \end{aligned}$$

Note that to obtain the general antiderivative of $F_1(x,y)$ with respect to x, we must add on an arbitrary function $g(y)$ depending on y alone, rather than the usual constant of integration. Similarly, we have

$$\begin{aligned} f(x,y) &= \int F_2(x,y)\,\mathrm{d}y \\ &= \int \left(x^2 + 3xy^2 + 2y\right)\mathrm{d}y \\ &= x^2y + xy^3 + y^2 + \underline{h(x)} \end{aligned}$$

The two expressions for $f(x,y)$ must be equal:

$$x^2y + xy^3 + g(y) = x^2y + xy^3 + y^2 + h(x)$$

This tells us that $\underline{g(y) = y^2}$ and $\underline{h(x) = 0}$, up to the addition of an arbitrary numerical constant C. Thus, we obtain the general potential function

$$f(x,y) = \boxed{x^2y + xy^3 + y^2 + C}$$

Example. Find a potential function for

$$\mathbf{F} = \left\langle 2xyz^{-1},\ z + x^2z^{-1},\ y - x^2yz^{-2} \right\rangle$$

Solution. If a potential function f exists, then it satisfies

$$f(x,y,z) = \int 2xyz^{-1}\,\mathrm{d}x = x^2yz^{-1} + f(y,z)$$
$$f(x,y,z) = \int \left(z + x^2z^{-1}\right)\mathrm{d}y = zy + x^2z^{-1}y + g(x,z)$$
$$f(x,y,z) = \int \left(y - x^2yz^{-2}\right)\mathrm{d}z = yz + x^2yz^{-1} + h(x,y)$$

These three ways of writing $f(x,y,z)$ must be equal:

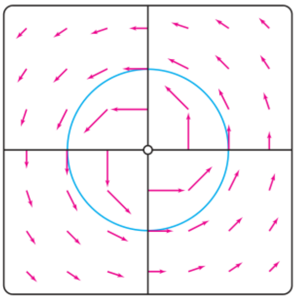
$$x^2yz^{-1} + f(y,z) = zy + x^2z^{-1}y + g(x,z) = yz + x^2yz^{-1} + h(x,y)$$

These equalities hold if $f(y,z) = yz$, $g(x,z) = 0$, and $h(x,y) = 0$. Thus, $\mathbf{F}$ is conservative and, for any constant C, a potential function is

$$f(x,y,z) = \boxed{x^2yz^{-1} + yz + C}$$

Remark. If $\mathbf{F}$ does not satisfy the cross partials property, then no potential function exists.

1.2 The Vortex Field



We continue with the example showing that there are vector fields which satisfy the cross partials property but are not conservative. Considering the vortex field:

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

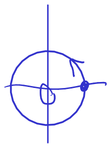
We check the cross partials property directly: $\frac{\partial}{\partial x}$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\left(x^2+y^2\right) - x\cdot\partial_x\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-\left(x^2+y^2\right) + y\cdot\partial_y\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$\oint_C F\,dr = 0$ for c.v.f.

Now consider the line integral of $\mathbf{F}$ around the unit circle $\mathcal{C}$ parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$:



$$F(\mathbf{r}(t))\cdot\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle = \sin^2 t + \cos^2 t = 1$$

$$\oint_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi \neq 0$$

If $\mathbf{F}$ were conservative, its circulation around every closed curve would be zero. Thus, $\mathbf{F}$ cannot be conservative, even though it satisfies the cross-partials condition.

This result does not contradict Theorem 3 because the domain of $\mathbf{F}$ does not satisfy the simply connected condition of the theorem. Because $\mathbf{F}$ is not defined at $(x,y) = (0,0)$, its domain is $\mathcal{D} = \{(x,y) \neq (0,0)\}$, and this domain is not simply connected.