Contents

1 §6.4: Green's Theorem 1
1.1 Area via Green's Theorem 4
1.2 Additivity of Circulation 5
1.3 More General Form of Green's Theorem 6
1.4 Flux Form of Green's Theorem 7
1.5 Remarks looking forward 8

---

1 §6.4: Green's Theorem

$\mathcal{C}$

Before stating Green's theorem, we need some notation. Consider a domain $\mathcal{D}$ whose boundary $\mathcal{C}$ is a **simple closed curve**—that is, a closed curve that does not intersect itself. Denote the boundary curve $\mathcal{C}$ by $\partial\mathcal{D}$. The **boundary orientation** of $\partial\mathcal{D}$ is counterclockwise.

Not simple:

Theorem 1 (Green's Theorem (circulation form))
Let $\mathcal{D}$ be a domain whose boundary $\partial\mathcal{D}$ is a simple closed curve, oriented counterclockwise. Then

$$\boxed{\oint_{\partial D} F_1\,\mathrm{d}x + F_2\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}A}$$

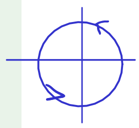
Recall that if $\mathbf{F}$ is a conservative vector field, (i.e., $\mathbf{F} = \boldsymbol{\nabla} f$) then the cross-partial property is satisfied:

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

In this case, Green's Theorem merely confirms what we already know: that the line integral of a conservative vector field around any closed curve is zero.

Example (Verifying Green's Theorem)**.** Verify Green's Theorem for the line integral along the unit circle $\mathcal{C}$, oriented counterclockwise:

$$\oint_{\mathcal{C}} xy^2\,\mathrm{d}x + x\,\mathrm{d}y$$



Solution. **First approach: Evaluate the line integral directly:**

Using the standard parametrization of the unit circle:

$$r(\theta) = \quad \begin{aligned} x &= \cos\theta, & y &= \sin\theta \\ \mathrm{d}x &= -\sin\theta\,\mathrm{d}\theta, & \mathrm{d}y &= \cos\theta\,\mathrm{d}\theta \end{aligned} \quad \theta \in [0, 2\pi]$$

The integrand in the line integral is

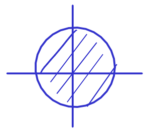
$$\begin{aligned} xy^2\,\mathrm{d}x + x\,\mathrm{d}y &= \cos\theta\sin^2\theta(-\sin\theta\,\mathrm{d}\theta) + \cos\theta(\cos\theta\,\mathrm{d}\theta) \\ &= \Big(-\cos\theta\sin^3\theta + \cos^2\theta\Big)\,\mathrm{d}\theta \end{aligned}$$

and

$$\begin{aligned} \oint_{\mathcal{C}} xy^2\,\mathrm{d}x + x\,\mathrm{d}y &= \int_0^{2\pi} \Big(-\cos\theta\sin^3\theta + \cos^2\theta\Big)\,\mathrm{d}\theta \\ &= -\left.\frac{\sin^4\theta}{4}\right|_0^{2\pi} + \left.\frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right|_0^{2\pi} \\ &= 0 + \frac{1}{2}(2\pi + 0) = \boxed{\pi} \end{aligned}$$

Second approach: Evaluate the line integral using Green's Theorem
In this example, $F_1 = xy^2$ and $F_2 = x$, so

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}xy^2 = 1 - 2xy$$



According to Green's Theorem,

$$\oint_{\mathcal{C}} xy^2\,\mathrm{d}x + x\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}A = \iint_{\mathcal{D}} (1 - 2xy)\,\mathrm{d}A$$

where $\mathcal{D}$ is the disk $x^2 + y^2 \leq 1$ enclosed by $\mathcal{C}$. The integral of $2xy$ over $\mathcal{D}$ is zero by symmetry—the contributions for positive and negative x cancel. We can check this directly:

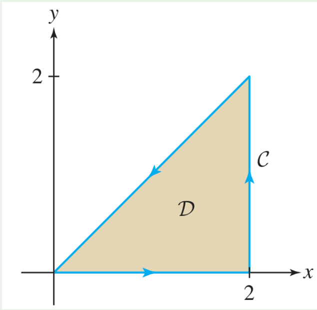
$$\iint_{\mathcal{D}} (-2xy)\,\mathrm{d}A = -2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\,\mathrm{d}y\,\mathrm{d}x = -\int_{-1}^{1} \underbrace{\left. xy^2 \right|_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}}_{\text{odd fctn wrt } x.}\,\mathrm{d}x = 0$$

Therefore,

$$\iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \iint_{\mathcal{D}} 1 \, \mathrm{d}A = \text{Area}(\mathcal{D}) = \boxed{\pi}$$

This agrees with the value obtained in the first approach. So Green's Theorem is verified in this case.

Example. Compute the circulation of $\mathbf{F}(x, y) = \langle \sin x, x^2 y^3 \rangle$ around the triangular path $\mathcal{C}$ with counterclockwise orientation shown below:



To compute the line integral directly, we would have to parametrize all three sides of the triangle. Instead, we apply Green's Theorem to the domain $\mathcal{D}$ enclosed by the triangle. This domain is described by $0 \le x \le 2$, $0 \le y \le x$.

Applying Green's Theorem, we obtain

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x} x^2 y^3 - \frac{\partial}{\partial y} \sin x = 2xy^3$$

$$\oint_{\mathcal{C}} \sin x \, \mathrm{d}x + x^2 y^3 \, \mathrm{d}y = \iint_{\mathcal{D}} 2xy^3 \, \mathrm{d}A = \int_0^2 \int_0^x 2xy^3 \, \mathrm{d}y \, \mathrm{d}x$$

$$= \int_0^2 \left(\frac{1}{2} xy^4 \Big|_0^x \right) \mathrm{d}x = \frac{1}{2} \int_0^2 x^5 \, \mathrm{d}x = \frac{1}{12} x^6 \Big|_0^2 = \boxed{\frac{16}{3}}$$

1.1 Area via Green's Theorem

In order to use Green's Theorem to compute the area of a domain $\mathcal{D}$ enclosing a simple closed curve $\mathcal{C}$, we want a vector field $\mathbf{F} = \langle F_1, F_2 \rangle$ such that $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 1$. Here are some possible options, which one can check satisfy the given relation:

- $\mathbf{F}(x, y) = \langle 0, x \rangle$
- $\mathbf{F}(x, y) = \langle -y, 0 \rangle$
- $\mathbf{F}(x, y) = \big\langle -y/2, x/2 \big\rangle$

Thus

$$\oint_{\mathcal{C}} F_1 \,\mathrm{d}x + F_2 \,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \iint_{\mathcal{D}} 1 \,\mathrm{d}A = \mathrm{area}(\mathcal{D})$$

Plugging in F_1 and F_2 for each of these three cases, we obtain the following three formulas for the area of the domain $\mathcal{D}$:

$$\boxed{\text{Area enclosed by } \mathcal{C} = \oint_{\mathcal{C}} x \,\mathrm{d}y = \oint_{\mathcal{C}} -y \,\mathrm{d}x = \frac{1}{2} \oint_{\mathcal{C}} x \,\mathrm{d}y - y \,\mathrm{d}x} \tag{1}$$

These formulas tell us how to compute an enclosed area by making measurements only along the boundary.

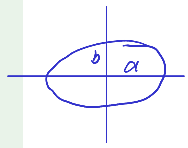
Example (Computing Area via Green's Theorem)**.** Compute the area of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ using a line integral.

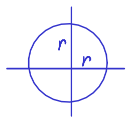
Solution. We parametrize the boundary of the ellipse by

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta < 2\pi$$

We calculate the area in each of the three possible ways. Using the first formula in (1):

$$\begin{aligned} \text{Enclosed area } &= \oint_{\mathcal{C}} x \,\mathrm{d}y = \int_0^{2\pi} (a\cos\theta)(b\cos\theta) \,\mathrm{d}\theta \\ &= ab \int_0^{2\pi} \cos^2\theta \,\mathrm{d}\theta \\ &= \boxed{\pi ab} \end{aligned}$$





$A = \pi r^2$

Using the second formula in (1), we get

$$\begin{aligned}\text{Enclosed area } &= \oint_{\mathcal{C}} -y\,\mathrm{d}x = \int_0^{2\pi} (-b\sin\theta)(-a\sin\theta)\,\mathrm{d}\theta \\ &= ab\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta \\ &= \pi ab\end{aligned}$$

And using the third formula in (1) yields

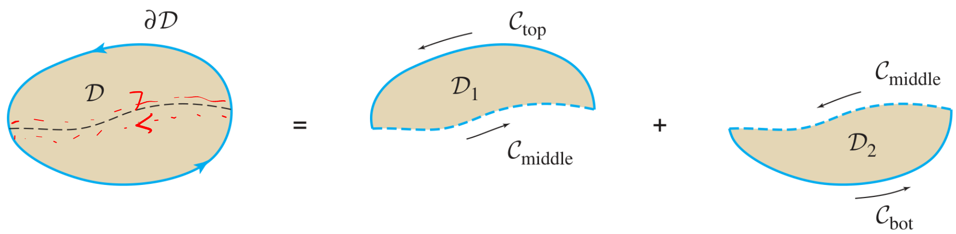
$$\begin{aligned}\text{Enclosed area } &= \frac{1}{2}\oint_{\mathcal{C}} x\,\mathrm{d}y - y\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{2\pi} [(a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)]\,\mathrm{d}\theta \\ &= \frac{ab}{2}\int_0^{2\pi} \left(\cos^2\theta + \sin^2\theta\right)\mathrm{d}\theta \\ &= \frac{ab}{2}\int_0^{2\pi} \mathrm{d}\theta = \pi ab\end{aligned}$$

All three methods yield the standard formula for the area of an ellipse.

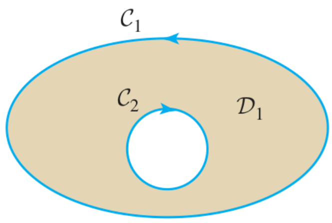
1.2 Additivity of Circulation

Circulation around a closed curve has an important additivity property: If we decompose a domain $\mathcal{D}$ into two (or more) nonoverlapping domains $\mathcal{D}_1$ and $\mathcal{D}_2$ that intersect only on part of their boundaries, then

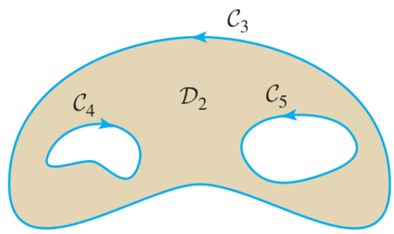
$$\oint_{\partial\mathcal{D}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint_{\partial\mathcal{D}_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} + \oint_{\partial\mathcal{D}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$



1.3 More General Form of Green's Theorem



(A) Oriented boundary of $\mathcal{D}_1$ is $\mathcal{C}_1 + \mathcal{C}_2$.



(B) Oriented boundary of $\mathcal{D}_2$ is $\mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_5$.

Consider a domain $\mathcal{D}$ whose boundary consists of more than one simple closed curve as in the above figure. As before, $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$ with its boundary orientation. In other words, the region lies to the left as the curve is traversed in the direction specified by the orientation. For the domains above,

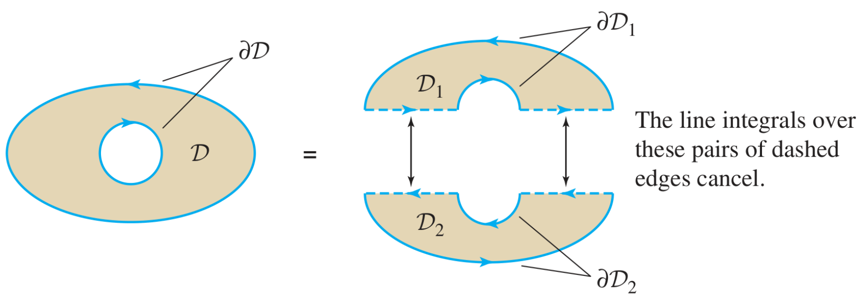
$$\underline{\partial\mathcal{D}_1 = \mathcal{C}_1 + \mathcal{C}_2}, \quad \underline{\partial\mathcal{D}_2 = \mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_5}$$

The curve $\mathcal{C}_5$ occurs with a minus sign because it is oriented counterclockwise, but the boundary orientation requires a clockwise orientation.

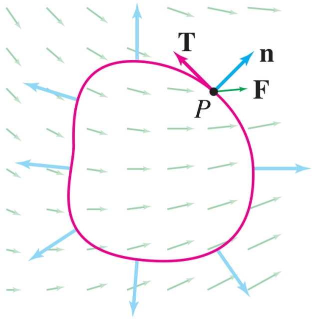
Green's Theorem remains valid for more general domains of this type:

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A$$

This equality is proved by decomposing $\mathcal{D}$ into smaller domains, each of which is bounded by a simple closed curve.



1.4 Flux Form of Green's Theorem



$$\oint \vec{F}\cdot d\vec{r} = \oint_C F_1\,dx + F_2\,dy$$
$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Suppose that we want to compute the flux of the vector field $\mathbf{F} = \langle F_1, F_2\rangle$ across the curve $\mathcal{C}$. That is, we want to integrate the normal component of $\mathbf{F}$ around the curve $\mathcal{C}$.

If the curve is parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$, the unit tangent vector is given by $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \dfrac{x'(t)}{\|\mathbf{r}'(t)\|}, \dfrac{y'(t)}{\|\mathbf{r}'(t)\|}\right\rangle$ and the outward unit normal vector is given by $\mathbf{n}(t) = \left\langle \dfrac{y'(t)}{\|\mathbf{r}'(t)\|}, \dfrac{-x'(t)}{\|\mathbf{r}'(t)\|}\right\rangle$ since its dot product with $\mathbf{T}$ is 0 and $\mathbf{n}$ points to the right as we travel around the curve.

The flux of $\mathbf{F}$ out of $\mathcal{C}$ is given by

$$\begin{aligned}
\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s &= \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,\|\mathbf{r}'(t)\|\,\mathrm{d}t \\
&= \int_a^b \left[\frac{F_1 y'(t)}{\|\mathbf{r}(t)\|} - \frac{F_2 x'(t)}{\|\mathbf{r}(t)\|}\right]\|\mathbf{r}(t)\|\,\mathrm{d}t \\
&= \int_a^b F_1 y'(t)\,\mathrm{d}t - F_2 x'(t)\,\mathrm{d}t \\
&= \int_a^b \underbrace{F_1}_{G_2}\,\mathrm{d}y - \underbrace{F_2}_{G_1}\,\mathrm{d}x \quad = \iint_D \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) dA
\end{aligned}$$

This is in a form to which we can apply Green's Theorem, but we have switched the roles of F_1 and F_2 and added a negative sign to the second term. So, Green's Theorem gives us

$$\boxed{\oint_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \int_{\partial\mathcal{D}} F_1\,\mathrm{d}y - F_2\,\mathrm{d}x = \iint_{\mathcal{D}} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)\mathrm{d}A}$$

1.5 Remarks looking forward

Divergence and curl are discussed in the next section. Once we have those operations defined, we are able to rewrite the the formulas for Green's theorem in the following way:

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \iint_{\mathcal{D}} \mathrm{curl}_z(\mathbf{F})\, \mathrm{d}A \qquad \text{(circulation form)}$$

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}s = \iint_{\mathcal{D}} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) \mathrm{d}A = \iint_{\mathcal{D}} \mathrm{div}(\mathbf{F})\, \mathrm{d}A \qquad \text{(flux form)}$$

It should also be mentioned that Green's theorem is the flat version of Stokes theorem. In other words, Stokes' theorem generalizes Green's theorem from regions lying on the xy-plane to surfaces.