Contents

1 §6.5: Divergence and Curl

Let $f(x, y, z)$ be a function of three variables, and $\mathbf{F} = \langle P, Q, R\rangle$. Setting

$$\nabla := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

we are able define the three main operations of vector calculus:

$$\begin{aligned}
\operatorname{grad} f &= \boldsymbol{\nabla} f &&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f &&= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\
\operatorname{curl} \mathbf{F} &= \boldsymbol{\nabla} \times \mathbf{F} &&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} &&= \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle \\
\operatorname{div} \mathbf{F} &= \boldsymbol{\nabla} \cdot \mathbf{F} &&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle &&= P_x + Q_y + R_z
\end{aligned}$$

Note that curl and divergence both operate on vector fields, but return different types of objects. Curl returns a vector field; divergence returns a scalar function.

Remark. Just as how vector fields that arise as the gradient of some potential function are sometimes called **gradient vector fields**, vector fields that arise as the curl of some vector field are somtimes called **curl vector fields**.

Linearity Both divergence and curl are **linear**:

$$\begin{aligned}
\operatorname{div}(\mathbf{F} + \mathbf{G}) &= \operatorname{div}(\mathbf{F}) + \operatorname{div}(\mathbf{G}) \\
\operatorname{div}(c\mathbf{F}) &= c \operatorname{div}(F)
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl}(\mathbf{F} + \mathbf{G}) &= \operatorname{curl}(\mathbf{F}) + \operatorname{curl}(\mathbf{G}) \\
\operatorname{curl}(c\mathbf{F}) &= c \operatorname{curl}(\mathbf{F})
\end{aligned}$$

Example. Compute the curl of $\mathbf{F} = \langle xy, e^x, y+z \rangle$.

Solution.

$\partial_x := \frac{\partial}{\partial x}$

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xy & e^x & y+z \end{vmatrix}$$
$$= \langle 1-0, -(0-0), e^x - x \rangle = \boxed{\langle 1, 0, e^x - x \rangle}$$

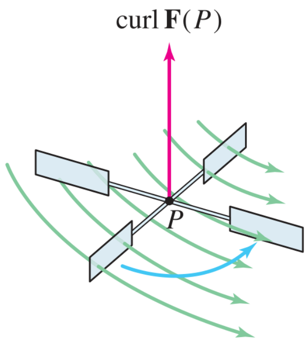
Example. Compute the divergence of $\mathbf{F} = \langle e^{xy}, xy, z^4 \rangle$.

Solution.

$$\frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial y} xy + \frac{\partial}{\partial z} z^4 = ye^{xy} + x + 4z^3$$

1.1 Physical interpretation of divergence and curl

Curl The magnitude of the vector curl $\mathbf{F}(P)$ is a measure of how fast the vector field $\mathbf{F}$, when considered as the velocity vector field of a fluid flow, would turn a paddle wheel inserted into the fluid.



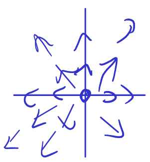
The paddle wheel is placed perpendicular to the vector curl $\mathbf{F}(P)$ in order to achieve the fastest rotation.

$\operatorname{div}(\vec{F})(P)$

Divergence Suppose $\mathbf{F}$ is the velocity vector field of a gas. When $\operatorname{div}(\mathbf{F}) > 0$ at a point P, an outflow of gas occurs near this point. In other words, the gas is expanding around this point, as might occur when the gas is heated. When $\operatorname{div}(\mathbf{F}) < 0$ at a point P, the gas is compressing toward P, as might occur when the gas is cooled. When $\operatorname{div}(\mathbf{F}) = 0$, the gas is neither compressing nor expanding near P.

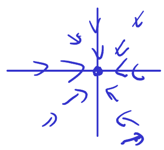
We call points where $\operatorname{div}(\mathbf{F}) > 0$ **sources** and points where $\operatorname{div}(\mathbf{F}) < 0$ **sinks**.

If a vector field has no sources or sinks, we say that the vector field is **incompressible**.



$\operatorname{div}(\vec{F}) > 0$

Source



$\operatorname{div}(F) < 0$

sink

Remark. Vector fields that satisfy $\operatorname{curl} \mathbf{F} = 0$ are sometimes called **curl-free** or **irrotational** vector fields.

Vector fields that satisfy $\operatorname{div} \mathbf{F} = 0$ are sometimes called **divergence-free**, or **incompressible** vector fields.

1.2 Reinterpreting conservative vector fields

Recall that conservative vector fields arise as gradients of potential functions ($\mathbf{F} = \boldsymbol{\nabla} f$), and that all conservative vector fields satisfy the cross-partials property:
If a vector field $\mathbf{F} = \langle \overset{x}{P}, \overset{y}{Q}, \overset{z}{R} \rangle$ is conservative, then

$$P_y = Q_x, \qquad Q_z = R_y, \qquad R_x = P_z$$

Now that we have curl, we can notice that the cross-partials property is equivalent to saying

$$\operatorname{curl} \mathbf{F} = \mathbf{0}.$$

Thus we have

Theorem 1 (OpenStax 6.17: Curl of a Conservative Vector Field)
If $\mathbf{F}$ is conservative, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

and

Theorem 2 (OpenStax 6.18: Curl Test for a Conservtive Field)
Let $\mathbf{F}$ be a vector field in space on a simply connected domain.
If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

1.3 Excursion to the abstract

Notice that the result about the curl of a conservative vector field is a rather general statement:

$$\operatorname{curl}(\boldsymbol{\nabla} f) = \mathbf{0}. \tag{1}$$

A similar statement can be found for the divergence of a curl vector field:

Theorem 3 (Divergence of the Curl)
Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$\operatorname{div} \operatorname{curl}(\mathbf{F}) = 0 \tag{2}$$

Proof.

$$\begin{aligned} \operatorname{div} \operatorname{curl} \mathbf{F} &= \boldsymbol{\nabla} \cdot \left\langle R_y - Q_z, P_z - R_x, Q_x - P_y \right\rangle \\ &= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0 \end{aligned}$$

□

We also know that

$$\operatorname{curl} \mathbf{F} = \mathbf{0} \implies \mathbf{F} = \boldsymbol{\nabla} f \quad \text{for some } f \tag{3}$$

There is an analogous result that says

$$\operatorname{div} \mathbf{F} = 0 \implies \mathbf{F} = \operatorname{curl} \mathbf{A} \quad \text{for some } \mathbf{A}. \tag{4}$$

Equations (1) to (4) relating curl, grad, and div can be combined into one statement, that the sequence

$$f \xrightarrow{\text{grad}} \mathbf{F} \xrightarrow{\text{curl}} \mathbf{G} \xrightarrow{\text{div}} g$$

is **exact**.

Other ways of writing this sequence are

$$\{\text{scalar fields on } U\} \xrightarrow{\text{grad}} \{\text{vector fields on } U\} \xrightarrow{\text{curl}} \{\text{vector fields on } U\} \xrightarrow{\text{div}} \{\text{scalar fields on } U\}$$

and

$$0 \to \mathbb{R} \hookrightarrow C^\infty(\mathbb{R}^3; \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(\mathbb{R}^3; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\mathbb{R}^3; \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\mathbb{R}^3; \mathbb{R}) \to 0$$

Generalizing this further would lead us to the de Rham cohomology, but that is far beyond the scope of this course ☺.