Contents

1 §6.6: Surface Integrals

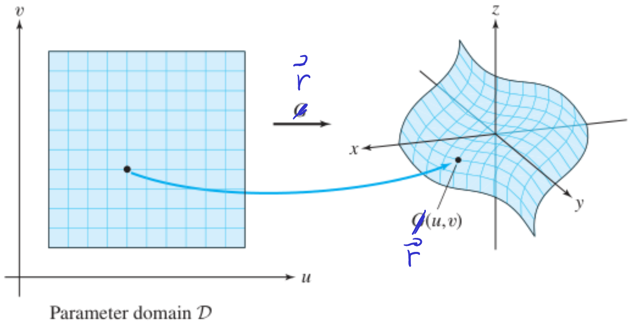
1.1 Parametrizing surfaces

A surface $\mathcal{S}$ may be parametrized in some form

$$\mathbf{r}(u,v) = \left\langle x(u,v), y(u,v), z(u,v) \right\rangle$$

The variables u, v (vary in a region $\mathcal{D}$) called the **parameter domain**. Two variables are needed to parameterize a surface because the surface is two-dimensional.

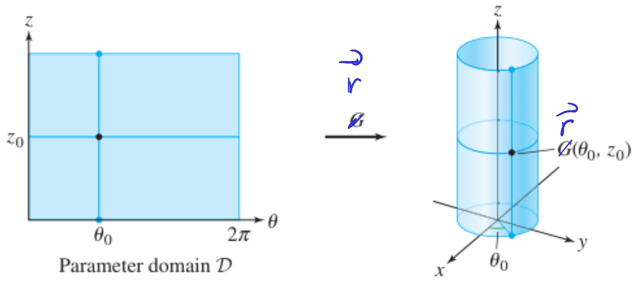
Figure 1: The parametric surface $\mathbf{r}(u,v) = \left\langle u+v, u^3 - v, v^3 - u \right\rangle$



1.2 Some useful parametrizations

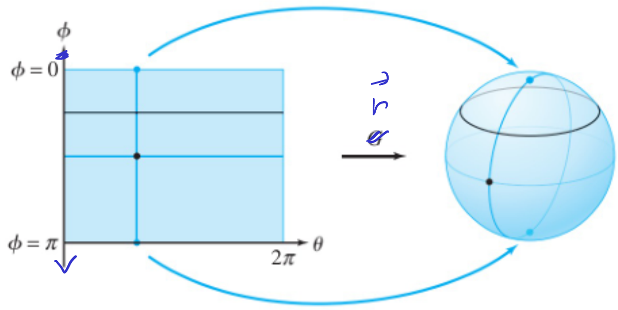
Parametrization of a cylinder:

$$\mathbf{r}(\theta, z) = \langle R\cos\theta, R\sin\theta, z\rangle\,, \quad 0 \le \theta < 2\pi, \quad -\infty < z < \infty$$



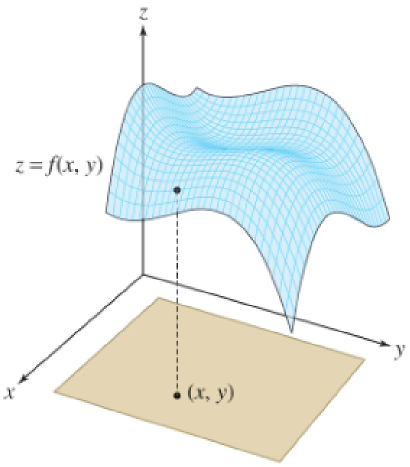
Parametriztion of a sphere:

$$\mathbf{r}(\theta, \varphi) = \langle R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi\rangle\,, \quad 0 \le \theta < 2\pi, \quad 0 \le \varphi \le \pi$$



Parametrization of a Graph $z = f(x, y)$**:**

$$\mathbf{r}(x, y) = \big\langle x, y, f(x, y)\big\rangle$$



1.3 Surface area of a parametric surface

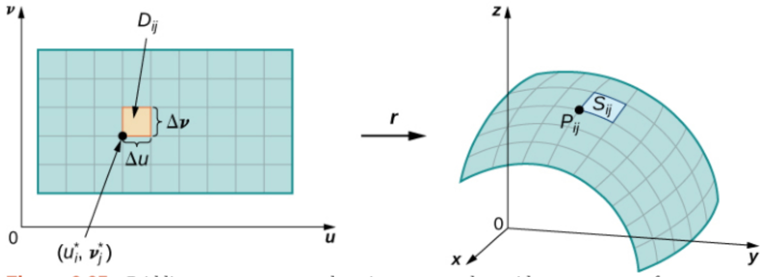


Figure 6.65 Grid lines on a parameter domain correspond to grid curves on a surface.

$t_u := \frac{\partial \vec{r}}{\partial u}$

$t_v := \frac{\partial r}{\partial v}$

For the grid curve $\mathbf{r}(u_i, v)$, the tangent vector at P_{ij} is

$$\mathbf{t}_v(P_{ij}) = \mathbf{r}_v(u_i, v_j) = \langle x_v(u_i, v_j), y_v(u_i, v_j), z_v(u_i, v_j) \rangle .$$

For the grid curve $\mathbf{r}(u, v_j)$, the tangent vector at P_{ij} is

$$\mathbf{t}_u(P_{ij}) = \mathbf{r}_u(u_i, v_j) = \langle x_u(u_i, v_j), y_u(u_i, v_j), z_u(u_i, v_j) \rangle .$$

If the piece S_{ij} is small enough, then the tangent plane at the point P_{ij} is a good approximation of the piece S_{ij}.

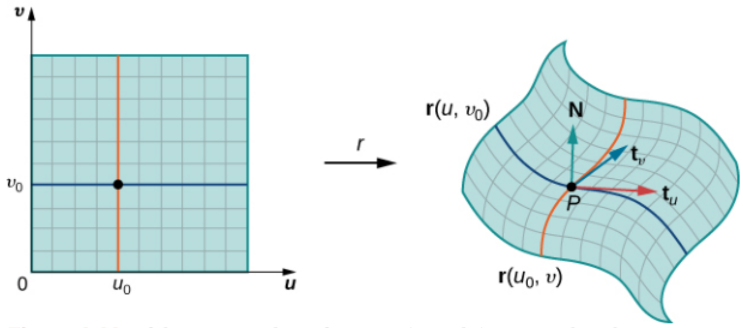


Figure 6.66 If the cross product of vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ exists, then there is a tangent plane.

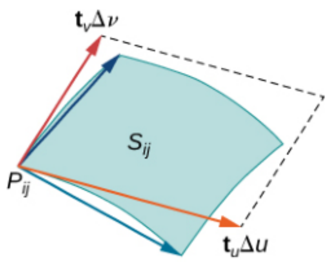


Figure 6.67 The parallelogram spanned by $\mathbf{t}_u$ and $\mathbf{t}_v$ approximates the piece of surface S_{ij}.

From this, we get the approximation

$$\Delta S_{ij} \approx \left\|\mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij})\right\|\Delta u \Delta v$$

The factor $\|\mathbf{t}_u \times \mathbf{t}_v\|$ is a distortion factor that measures how the area of a small rectangle D_{ij} is altered under the mapping $\mathbf{r}(u, v)$. Note that this behaves like the Jacobian.

We have the following formula for the surface area of S:

$$\text{Area}(S) = \iint_D \|\mathbf{t}_u \times \mathbf{t}_v\| \, \mathrm{d}u \, \mathrm{d}v$$

Example (Openstax Example 6.63)**.** Verify that the surface area of a sphere $x^2 + y^2 + z^2 = r^2$ is $4\pi r^2$

Solution. The sphere has parameterization

$$\langle r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi\rangle, \quad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi$$

The tangent vectors are

$$\mathbf{t}_\theta = \langle -r\sin\theta\sin\phi,\ r\cos\theta\sin\phi,\ 0\rangle \text{ and } \mathbf{t}_\phi = \langle r\cos\theta\cos\phi,\ r\sin\theta\cos\phi,\ -r\sin\phi\rangle.$$

Therefore,

$$\begin{aligned}\mathbf{t}_\phi \times \mathbf{t}_\theta &= \left\langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin^2\theta\sin\phi\cos\phi + r^2\cos^2\theta\sin\phi\cos\phi\right\rangle \\ &= \left\langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin\phi\cos\phi\right\rangle.\end{aligned}$$

Now,

$$\begin{aligned}\left\|\mathbf{t}_\phi \times \mathbf{t}_\theta\right\| &= \sqrt{r^4\sin^4\phi\cos^2\theta + r^4\sin^4\phi\sin^2\theta + r^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\cos^2\phi} \\ &= r^2\sqrt{\sin^2\phi} \\ &= r^2\sin\phi.\end{aligned}$$

$\sqrt{x^2} = |x|$

Notice that $\sin\phi \ge 0$ on the parameter domain because $0 \le \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is therefore given by

$$\int_0^{2\pi}\int_0^{\pi} r^2\sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta = r^2\int_0^{2\pi} 2 \, \mathrm{d}\theta = 4\pi r^2$$

1.4 Scalar surface integral

The **surface integral of a scalar-valued function** f over a piecewise smooth curve S is defined as

$$\iint_S f(x,y,z)\,\mathrm{d}S = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}$$

In the limit, this formula becomes

$$\boxed{\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))\|\mathbf{t}_u \times \mathbf{t}_v\|\,\mathrm{d}A}$$

This equation allows us to calculate a surface integral by transforming it into a double integral. Note that when $f(x,y,z)=1$, the formula computes the surface area of S.

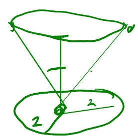
It is analogous to the formula for line integrals from §6.2:

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\left\|\mathbf{r}'(t)\right\|\mathrm{d}t$$

Example. Calculate the surface area of the portion S of the cone $x^2+y^2=z^2$ lying above the disk $x^2+y^2 \le 4$. Then calculate $\iint_S x^2 z\,\mathrm{d}S$.

Solution. The cone is given by $z^2=r^2$, so we can parametrize the surface S as

$$\mathbf{r}(\theta,u) = \langle u\cos\theta, u\sin\theta, u\rangle \qquad 0 \le u \le 2, \quad 0 \le \theta < 2\pi$$



We compute

$$\begin{aligned}
\mathbf{t}_\theta &= \langle -u\sin\theta, u\cos\theta, 0\rangle \\
\mathbf{t}_u &= \langle \cos\theta, \sin\theta, 1\rangle \\
\mathbf{t}_\theta \times \mathbf{t}_u &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -u\sin\theta & u\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \langle u\cos\theta, u\sin\theta, -u\rangle \\
\|\mathbf{t}_\theta \times \mathbf{t}_u\| &= \sqrt{u^2\cos^2\theta + u^2\sin^2\theta + (-u)^2} = \sqrt{2u^2} = \sqrt{2}|u|
\end{aligned}$$

Since $u \ge 0$ on our domain, we drop the absolute value.
Computing the surface area:

$$\text{Area}(S) = \iint_D \|\mathbf{t}_\theta \times \mathbf{t}_u\|\,\mathrm{d}S = \int_0^2 \int_0^{2\pi} \sqrt{2}u\,\mathrm{d}\theta\,\mathrm{d}u = \sqrt{2}\pi\,u^2\Big|_0^2 = \boxed{4\sqrt{2}\pi}$$

Computing the surface integral: We rewrite $f(x,y,z)=x^2z$ in terms of the parameters u,θ and evaluate:

$$f(\mathbf{r}(\theta,u)) = f(u\cos\theta, u\sin\theta, u) = (u\cos\theta)^2 u = u^3\cos^2\theta$$

$$\begin{aligned}
\iint_S f(x,y,z)\,\mathrm{d}S &= \int_0^2 \int_0^{2\pi} f(\mathbf{r}(\theta,u))\|\mathbf{t}_\theta \times \mathbf{t}_u\|\,\mathrm{d}\theta\,\mathrm{d}u \\
&= \int_0^2 \int_0^{2\pi} (u^3\cos^2\theta)(\sqrt{2}u)\,\mathrm{d}\theta\,\mathrm{d}u \\
&= \sqrt{2}\int_0^2 u^4\,\mathrm{d}u \cdot \int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta \\
&= \sqrt{2}\frac{32}{5}\pi = \boxed{\frac{32\sqrt{2}\pi}{5}}
\end{aligned}$$

1.5 Orientation of a surface

In order to define the vector surface integral (surface integral over a vector field) the orientation matters. This orientation is determined by a choice of unit normal vector.

If it is possible to choose a unit normal vector $\mathbf{N}$ at every point (x, y, z) on S so that $\mathbf{N}$ varies continuously over S, then S is **orientable**. Such a choice of unit normal vector at each point gives the **orientation of a surface** S.

Informally, a choice of orientation gives S an “outer” side and an “inner” side (or an “upward” side and a “downward” side), just as a choice of orientation of a curve gives the curve “forward” and “backward” directions.

Let S be a smooth orientable surface with parameterization $\mathbf{r}(u, v)$. For each point $\mathbf{r}(a, b)$ on the surface, vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ lie in the tangent plane at that point. Vector $\mathbf{t}_u \times \mathbf{t}_v$ is normal to the tangent plane at $\mathbf{r}(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\mathbf{N} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|}$$

gives an orientation of surface S.

Remark. The typical convention is to use *outward* pointing normal vectors. What is considered “outward” depends on context (the surface in question).

Orientation will need to be considered when setting up the integrals for Stokes theorem and Divergence theorem in the following sections.

1.6 Vector surface integral

The definition of the **surface integral of a vector field** is analogous to the definition of the flux of a vector field along a plane curve.

$\int_C \vec{F} \cdot d\vec{r}$

$\int_C \vec{F} \cdot \vec{N}\, ds$

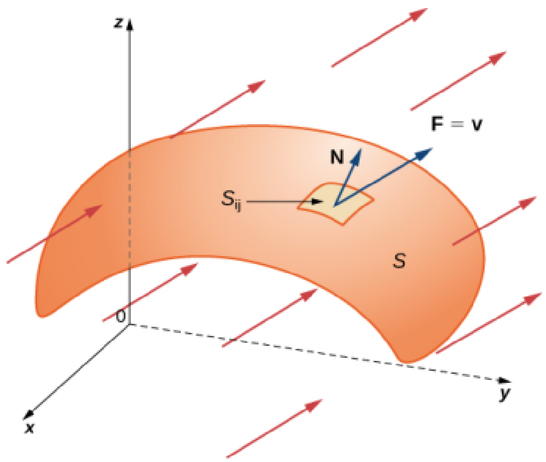


Figure 6.78 The mass of fluid per unit time flowing across S_{ij} in the direction of $\mathbf{N}$ can be approximated by $(\rho\mathbf{v}\cdot\mathbf{N})\Delta S_{ij}$.

Definition. Let $\mathbf{F}$ be a continuous vector field with a domain that contains an oriented surface S with unit normal vector $\mathbf{N}$. The **surface integral** of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{N}\,\mathrm{d}S$$

Let $\mathbf{r}(u,v)$ be a parametrization of S with parameter domain D. Then

$$\begin{aligned}
\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \iint_S \mathbf{F}\cdot\mathbf{N}\,\mathrm{d}S \\
&= \iint_S \mathbf{F}\cdot\frac{\mathbf{t}_u\times\mathbf{t}_v}{\|\mathbf{t}_u\times\mathbf{t}_v\|}\,\mathrm{d}S \\
&= \iint_D \left(\mathbf{F}(\mathbf{r}(u,v))\cdot\frac{\mathbf{t}_u\times\mathbf{t}_v}{\|\mathbf{t}_u\times\mathbf{t}_v\|}\right)\|\mathbf{t}_u\times\mathbf{t}_v\|\,\mathrm{d}A \\
&= \iint_D \big(\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{t}_u\times\mathbf{t}_v)\big)\,\mathrm{d}A
\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\boxed{\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_D \big(\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{t}_u\times\mathbf{t}_v)\big)\,\mathrm{d}A}$$

Example (OpenStax Example 6.70)**.** Calculate the surface integral $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F} = \langle -y, x, 0 \rangle$ and S is the surface with parameterization

$$\mathbf{r}(u,v) = \left\langle u, v^2 - u, u + v \right\rangle, \qquad 0 \le u < 3, \quad 0 \le v \le 4$$

Solution. The tangent vectors are $\mathbf{t}_u = \langle 1, -1, 1 \rangle$ and $\mathbf{t}_v = \langle 0, 2v, 1 \rangle$. Therefore,

$$\mathbf{t}_u \times \mathbf{t}_v = \langle -1 - 2v, -1, 2v \rangle.$$

~~By Equation 6.21,~~

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_0^4 \int_0^3 \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^4 \int_0^3 \left\langle u - v^2, u, 0 \right\rangle \cdot \langle -1 - 2v, -1, 2v \rangle\, \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^4 \int_0^3 \left[(u - v^2)(-1 - 2v) - u \right] \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\, \mathrm{d}u\, \mathrm{d}v \\
&= \int_0^4 \left[2v^3 u + v^2 u - vu^2 - u^2 \right]_0^3 \mathrm{d}v \\
&= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, \mathrm{d}v \\
&= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v \right]_0^4 \\
&= 340
\end{aligned}$$

Therefore, the flux of $\mathbf{F}$ across S is 340.