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1 §6.7: Stokes Theorem

Stokes' theorem is a higher-dimensional generalization of Green's Theorem. It relates a vector surface integral over a surface S in space to a line integral along its boundary ∂S.

To get signs correct for Stokes' theorem, we want the direction of the normal vectors along the surface S to match up with the orientation of the boundary C. They match up if they satisfy the right-hand rule. The OpenStax text calls this "positive orientation".

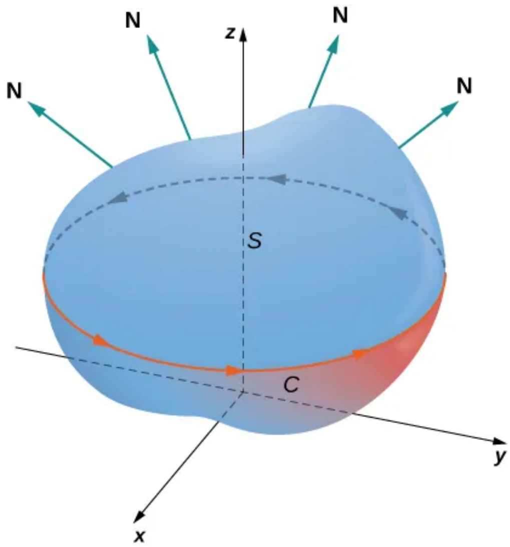
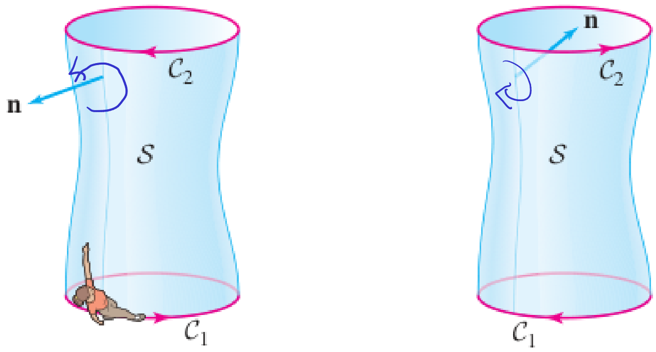


Figure 6.79 Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.



(A) (B)

> **Theorem 1** (Stokes' Theorem)
>
> Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation. If $\mathbf{F}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then
>
> $$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}}$$

Remark. This can also be written as

$$\boxed{\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}}$$

$F = \langle P, Q, R \rangle$

$= \iint_S \underbrace{\operatorname{curl} F \cdot \vec{N}}_{= Q_x - P_y} \, dS \qquad \vec{N} = \langle 0, 0, 1 \rangle = \hat{k}$

Remark. If the surface S is a flat region in the xy-plane, then Stokes' theorem reduces to Green's theorem.

Remark. The curl measures the extent to which $\mathbf{F}$ fails to be conservative. If $\mathbf{F}$ is conservative, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and Stokes' theorem merely confirms what we already know: The circulation of a conservative vector field around a closed path is zero.

$$\operatorname{curl} \mathbf{F} = \mathbf{0} \implies \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Example (Openstax 6.73)**.** Verify that Stokes' theorem is true for the vector field $\mathbf{F} = \langle y, 2z, x^2 \rangle$ and surface S, where S is the paraboloid $z = 4 - x^2 - y^2$. Assume the surface is outward oriented and $z \geq 0$.

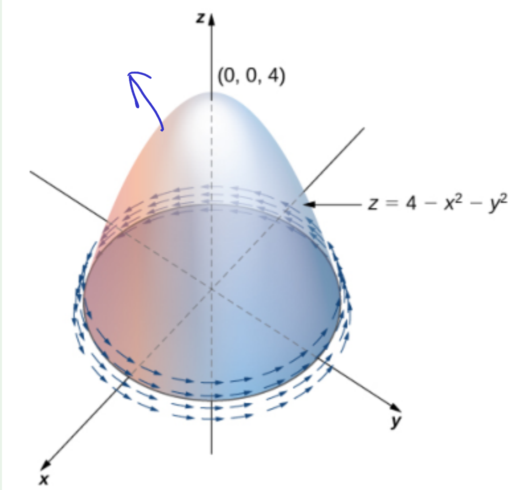
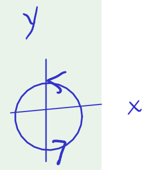


Figure 6.83 Verifying Stokes' theorem for a hemisphere in a vector field.



Solution. First, we compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D (\operatorname{curl} \mathbf{F})(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, \mathrm{d}A$$

For this surface, we have

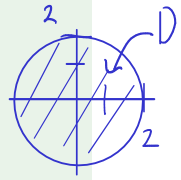
$$\begin{aligned} \mathbf{r}(x, y) &= \langle x, y, 4 - x^2 - y^2 \rangle \\ \mathbf{t}_x = \mathbf{r}_x &= \langle 1, 0, -2x \rangle \\ \mathbf{t}_{\cancel{x}\,y} = \mathbf{r}_{\cancel{x}\,y} &= \langle 0, 1, -2y \rangle \\ \mathbf{t}_x \times \mathbf{t}_y &= \langle 2x, 2y, 1 \rangle \end{aligned}$$

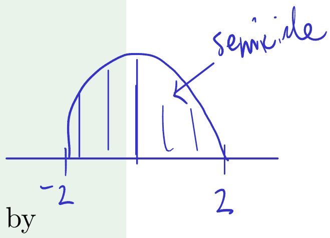
This normal vector is pointed in the correct direction (outward). Also,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & x^2 \end{vmatrix} = \langle -2, -2x, -1 \rangle$$

Thus, the surface integral can be computed as

$$\begin{aligned}
\iint_S \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \iint_D (\operatorname{curl}\mathbf{F})(\mathbf{r}(u,v))\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,\mathrm{d}A\\
&= \iint_D \langle -2,-2x,-1\rangle\cdot\langle 2x,2y,1\rangle\,\mathrm{d}A\\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(-4x-4xy-1)\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_{-2}^{2}\left(\cancelto{0}{-8x\sqrt{4-x^2}}-0-2\sqrt{4-x^2}\right)\mathrm{d}x\\
&= -2\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x = -2\cdot\frac{1}{2}\pi 2^2 = \boxed{-4\pi}
\end{aligned}$$





Next we compute the line integral around $C=\partial S$. It can be parametrized by

$$\mathbf{r}(t)=\langle \overset{x}{2\cos t}, \overset{y}{2\sin t}, \overset{z}{0}\rangle \quad t\in[0,2\pi]$$

so

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t\\
&= \int_0^{2\pi}\langle 2\sin t, 0, 4\cos^2 t\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle\,\mathrm{d}t\\
&= \int_0^{2\pi} -4\sin^2 t\,\mathrm{d}t = \boxed{-4\pi}
\end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.

Example (Openstax 6.74)**.** Calculate the surface integral $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathrm{d}\mathbf{S}$, where S is the surface illustrated below, oriented outward, and $\mathbf{F}=\langle z, 2xy, x+y\rangle$.

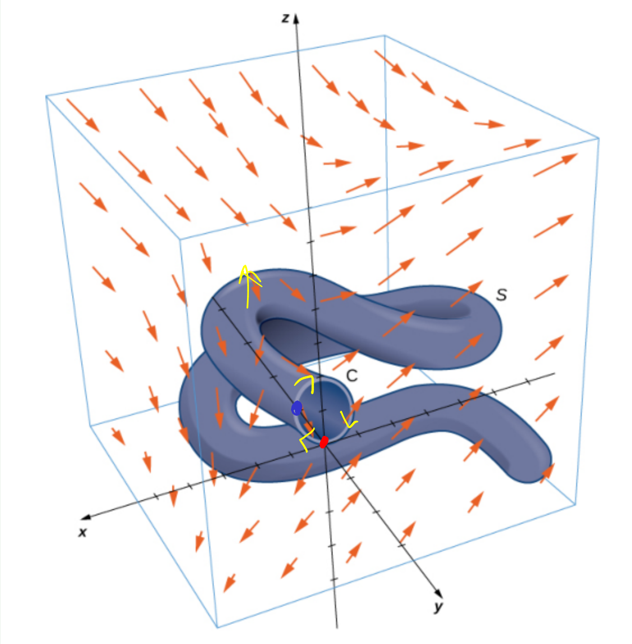


Figure 6.84 A complicated surface in a vector field.

Solution. Since by Stokes theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

instead of integrating over the surface, which is difficult, we can integrate over the boundary C, which is a circle of radius 1. For the correct orientation, C must be parametrized clockwise from the perspective given, which can be done with the parametrization

$$\mathbf{r}(t) = \langle \sin t, 0, 1 - \cos t \rangle, \qquad 0 \le t \le 2\pi$$

$t = 0 : (0, 0, 0)$

$t = \frac{\pi}{2} : (1, 0, 1)$

The line integral is thus

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} \langle 1 - \cos t, 0, \sin t \rangle \cdot \langle \cos t, 0, \sin t \rangle\, dt \\ &= \int_0^{2\pi} (\cos t - \cos^2 t + \sin^2 t)\, dt \\ &= \int_0^{2\pi} (\cos t - \cos(2t))\, dt \\ &= \boxed{0} \end{aligned}$$

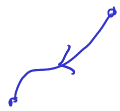
$\cos(2t) = \cos^2 t - \sin^2 t$

Remark. Stokes' theorem says that the surface integral only depends on the line integral around the boundary. That means if two surfaces S_1, S_2 have the same boundary C and the same orientation, then by Stokes' theorem

$$\iint_{S_1} \underline{\operatorname{curl} \mathbf{F}} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \underline{\operatorname{curl} \mathbf{F}} \cdot d\mathbf{S}$$

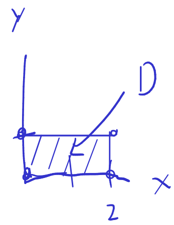
Which means that we could have computed $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ in the previous example by instead simplifying the surface to, say, the disk enclosed by the boundary curve C, and integrating on that simplified surface.

This shows that flux integrals of curl vector fields are **surface independent** in the same way that line integrals of gradient fields are path independent.



Example (OpenStax 6.75). Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, x^2 + y^2 + z^2, yz \rangle$ and C is the boundary of the parallelogram with vertices $(0, 0, 1)$, $(0, 1, 0)$, $(2, 0, -1)$, and $(2, 1, -2)$.

Solution. Calculating this line integral directly would require parametrizing all four sides of this parallelogram, calculating four separate line integrals, and adding the result. Instead, we opt to use Stokes' theorem. Let S be the surface of the parallelogram. It is a portion of the graph $z = 1 - x - y$ varying over the rectangle D with vertices $(0, 0)$, $(0, 1)$, $(2, 0)$, $(2, 1)$. This gives a parametrization of S:



$$\mathbf{r}(x, y) = \langle x, y, 1 - x - y \rangle \qquad x \in [0, 2], \quad y \in [0, 1]$$

We compute

$$\begin{aligned}
\mathbf{t}_x &= \langle 1, 0, -1 \rangle \\
\mathbf{t}_y &= \langle 0, 1, -1 \rangle \\
\mathbf{t}_x \times \mathbf{t}_y &= \langle 1, 1, 1 \rangle \\
\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xy & x^2 + y^2 + z^2 & yz \end{vmatrix} = \langle -z, 0, x \rangle
\end{aligned}$$

Thus Stokes' theorem gives

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} \\
&= \iint_D (\operatorname{curl} \mathbf{F})(\mathbf{r}(x,y)) \cdot (\mathbf{t}_x \times \mathbf{t}_y)\, \mathrm{d}A \\
&= \int_0^2 \int_0^1 \langle -1 + x + y, 0, x \rangle \cdot \langle 1, 1, 1 \rangle\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_0^2 \int_0^1 (2x + y - 1)\, \mathrm{d}y\, \mathrm{d}x \\
&= \boxed{3}
\end{aligned}$$

2 Homework hints

6.333 To parametrize C, you'll need to complete the square to recognize what C is when projected onto the xy-plane. Computing the integral is a bit involved. The answer you should get is $\boxed{0}$

6.344 Orientation matters somewhat for this. The integral is easy to compute, but the setup is easy to mess up. Parametrize the plane that the triangle lives on and use that parametrization to compute the normal vector in the double integral. You should get $\boxed{-18}$

6.346 The integral should be straightforward. Line integral over an ellipse. You should get $\boxed{0}$