Contents

1 §6.8: The Divergence Theorem

1.1 Overview of Theorems

Fundamental Theorem of Calculus

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(P_1) - f(P_0)$$

Green's theorem, circulation form

$$\iint_D (Q_x - P_y)\,\mathrm{d}A = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

Since $Q_x - P_y = \operatorname{curl}\mathbf{F}\cdot\mathbf{k}$ and curl is a derivative of sorts, Green's theorem relates the integral of derivative $\operatorname{curl}\mathbf{F}$ over the planar region D to an integral of $\mathbf{F}$ over the boundary of D.

Green's theorem, flux form

$$\iint_D (P_x + Q_y)\,\mathrm{d}A = \int_C \mathbf{F}\cdot\mathbf{N}\,\mathrm{d}s$$

Since $P_x + Q_y = \operatorname{div}\mathbf{F}$ and divergence is a derivative of sorts, the flux form of Green's theorem relates the integral of derivative $\operatorname{div}\mathbf{F}$ over the planar region D to an integral of $\mathbf{F}$ over the boundary of D.

Stokes' theorem

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative $\operatorname{curl}\mathbf{F}$ over the surface S (not necessarily planar) to an integral of $\mathbf{F}$ over the boundary of S.

$$\int_M f\,d\omega = \int_{\partial M} f\,\omega$$

Theorem 1 (The Divergence Theorem)
Let S be a piecewise, smooth closed surface that encloses a solid E in space. Assume that S is oriented outward, and let $\mathbf{F}$ be a vector field with continuous partial derivatives on an open region containing E. Then

$$\iiint_E \operatorname{div} \mathbf{F}\, \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

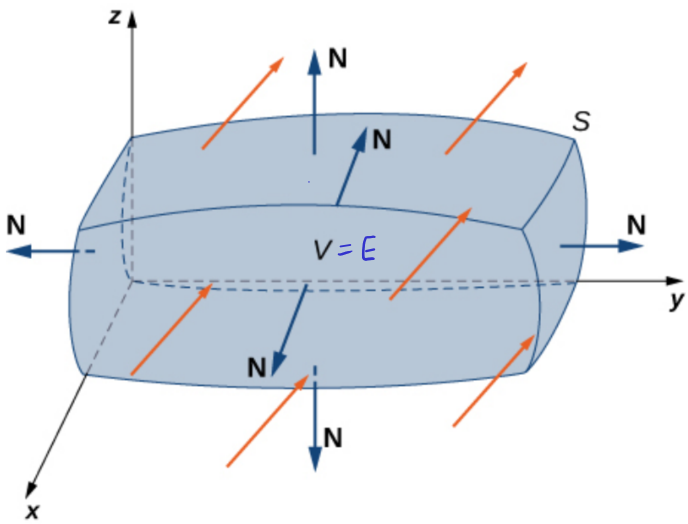
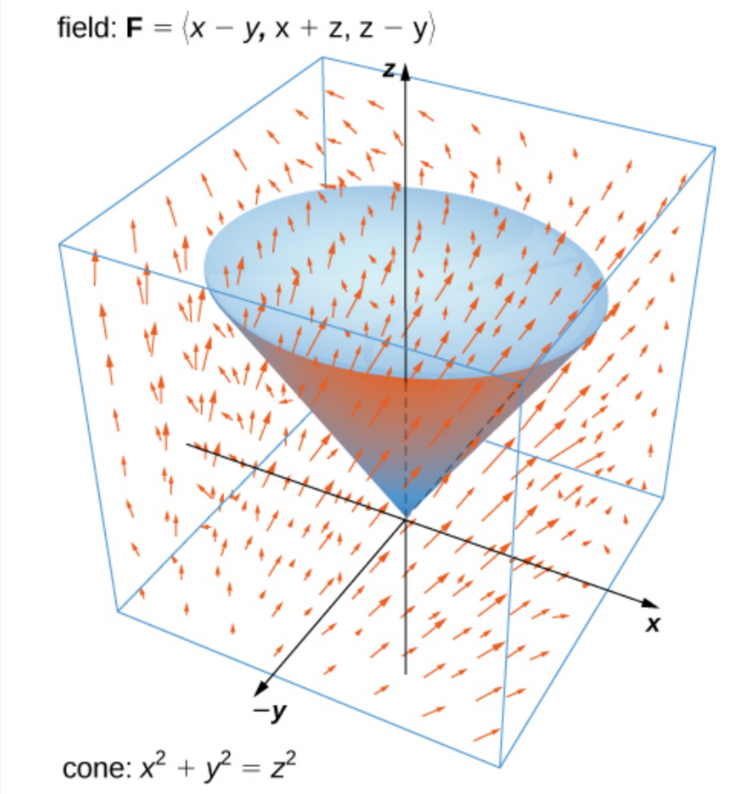


Figure 6.87 The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Remark. The divergence theorem is Green's theorem (flux version) in one higher dimension.

Example (OpenStax 6.77)**.** Verify the divergence theorem for the vector field $\mathbf{F} = \langle x - y, x + z, z - y \rangle$ and the surface S that consists of the cone $x^2 + y^2 = z^2$, $0 \leq z \leq 1$, and the circular top of the cone. Assume this surface is positively oriented.



Solution. Let E be the solid cone enclosed by S. We verify that

$$\iiint_E \operatorname{div} \mathbf{F}\, \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

First we compute the triple integral. Note that $\operatorname{div} \mathbf{F} = 2$, thus

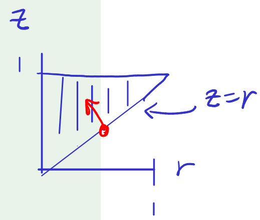
$$\iiint_E \operatorname{div} \mathbf{F}\, \mathrm{d}V = 2 \iiint_E \mathrm{d}V = 2(\text{volume of } E) = 2\frac{1}{3}\pi(1)^2(1) = \boxed{\frac{2\pi}{3}}$$

Now we compute the flux integral. We break S into two pieces: the circular top of the cone and the lateral surface of the cone. Call them S_1 and S_2, respectively. To compute the flux across the circular top S_1, we use

$$\begin{aligned}
\mathbf{r}_1(u,v) &= \langle u\cos v, u\sin v, 1\rangle \qquad u\in[0,1], \quad v\in[0,2\pi] \\
\mathbf{t}_u &= \langle \cos v, \sin v, 0\rangle \\
\mathbf{t}_v &= \langle -u\sin v, u\cos v, 0\rangle \\
\mathbf{t}_u\times\mathbf{t}_v &= \langle 0,0,u\rangle \\
\mathbf{F}(\mathbf{r}_1(u,v)) &= \langle u\cos v - u\sin v, u\cos v - 1, 1 - u\sin v\rangle \\
\mathbf{F}(\mathbf{r}_1(u,v))\cdot(\mathbf{t}_u\times\mathbf{t}_v) &= u - u^2\sin v
\end{aligned}$$

so the flux across S_1 is

$$\begin{aligned}
\iint_{S_1}\mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \int_0^1\int_0^{2\pi}\mathbf{F}(\mathbf{r}_1(u,v))\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,\mathrm{d}A \\
&= \int_0^1\int_0^{2\pi}(u - \cancelto{0}{u^2\sin v})\,\mathrm{d}v\,\mathrm{d}u \\
&= \int_0^1 2\pi u\,\mathrm{d}u = \boxed{\pi}
\end{aligned}$$



To compute the flux over the lateral surface S_2, we use

$$\begin{aligned}
\mathbf{r}_2(u,v) &= \langle u\cos v, u\sin v, u\rangle, \qquad u\in[0,1], \quad v\in[0,2\pi] \\
\mathbf{t}_u &= \langle \cos v, \sin v, 1\rangle \\
\mathbf{t}_v &= \langle -u\sin v, u\cos v, 0\rangle \\
\mathbf{t}_u\times\mathbf{t}_v &= \langle -u\cos v, -u\sin v, u\rangle
\end{aligned}$$

Notice that this is inward pointing, so we instead use $\mathbf{t}_v\times\mathbf{t}_u = \langle u\cos v, u\sin v, -u\rangle$

$$\begin{aligned}
\mathbf{F}(\mathbf{r}_2(u,v)) &= \langle u\cos v - u\sin v,\ u\cos v + u,\ u - u\sin v\rangle \\
\mathbf{F}(\mathbf{r}_2(u,v))\cdot(\mathbf{t}_v\times\mathbf{t}_u) &= u^2(\cos^2 v - \sin v\cos v) + u^2(\sin v\cos v + \sin v) - u^2(1-\sin v) \\
&= u^2\left[\cos^2 v - \sin v\cos v + \sin v\cos v + \sin v - 1 + \sin v\right] \\
&= u^2\left[\cos^2 v + 2\sin v - 1\right]
\end{aligned}$$

so the flux across S_2 is

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_0^1 \int_0^{2\pi} \mathbf{F}(\mathbf{r}_2(u,v)) \cdot (\mathbf{t}_v \times \mathbf{t}_u)\,\mathrm{d}A \\ &= \int_0^1 \int_0^{2\pi} u^2(\cos^2 v + 2\sin v - 1)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_0^1 u^2\,\mathrm{d}u \cdot \int_0^{2\pi} (\cos^2 v + 2\sin v - 1)\,\mathrm{d}v \\ &= \frac{1}{3} \cdot (\pi + 0 - 2\pi) = \boxed{-\frac{\pi}{3}}\end{aligned}$$

The total flux across S is

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \pi + \left(-\frac{\pi}{3}\right) = \boxed{\frac{2\pi}{3}}$$

Thus we have verified the divergence theorem for this example.

1.2 Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface S and a triple integral over the solid enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example (OpenStax 6.78)**.** Calculate the surface integral $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where S is cylinder $x^2 + y^2 = 1$, $0 \le z \le 2$, including the circular top and bottom, and $\mathbf{F} = \left\langle \frac{x^3}{3} + yz, \frac{y^3}{3} - \sin(xz), z - x - y \right\rangle$.

Solution. Computing this integral without the divergence theorem would require breaking the cylinder into three parts, and computing three surface integrals.

By contrast, the divergence theorem allows us to calculate the single triple integral $\iiint_E \operatorname{div} \mathbf{F}\,\mathrm{d}V$, where E is the solid enclosed by the cylinder. Using the divergence theorem and converting to cylindrical coordinates, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,\mathrm{d}V\\ &= \iiint_E (x^2+y^2+1)\,\mathrm{d}V\\ &= \int_0^{2\pi}\int_0^1\int_0^2 (r^2+1)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta\\ &= 4\pi\int_0^1 (r^2+1)r\,\mathrm{d}r\\ &= 4\pi\left[\frac{1}{4}r^4+\frac{1}{2}r^2\right]_0^1 = 4\pi\cdot\frac{3}{4} = \boxed{3\pi}\end{aligned}$$

Example (Openstax 6.79)**.** Let $\mathbf{v} = \left\langle -\frac{y}{z}, \frac{x}{z}, 0\right\rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4$, $2 \le y \le 5$, $1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.

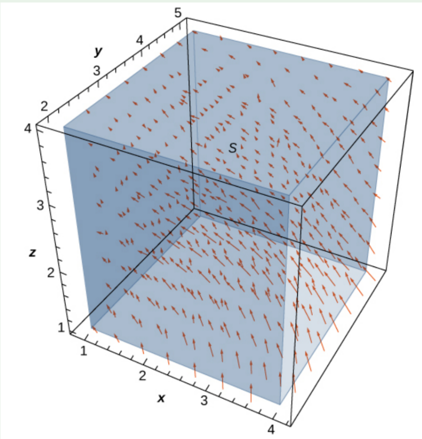


Figure 6.90 Vector field $\mathbf{v} = \langle -\frac{y}{z}, \frac{x}{z}, 0\rangle$.

Solution. The flow rate of fluid across S is $\iint_S \mathbf{v}\cdot \mathrm{d}\mathbf{S}$. Calculating this flux integral direction would require six separate flux integral, one for each face of the cube. Instead, we can use the divergence theorem:

$$\begin{aligned}\iint_S \mathbf{v}\cdot \mathrm{d}\mathbf{S} &= \iiint_C \operatorname{div}(\mathbf{v})\,\mathrm{d}V\\ &= \iiint_C 0\,\mathrm{d}V = \boxed{0}\end{aligned}$$

$\vec{F}(x,y,z) = \langle f(y,z), g(x,z), h(x,y)\rangle$

$\operatorname{div}\vec{F} = 0$