1. (a) $\vec{r}(t) = (\sin(2t), \cos(2t), 7)$
$\vec{r}'(t) = (2\cos(2t), -2\sin(2t), 0)$
$\vec{r}''(t) = (-4\sin(2t), -4\cos(2t), 0)$

Unit Tangent Vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{(2\cos(2t), -2\sin(2t), 0)}{\sqrt{4\cos^2(2t) + 4\sin^2(2t)}}$

$= \underline{(\cos(2t), -\sin(2t), 0)}$

Unit Normal

$\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \dfrac{(-2\sin(2t), -2\cos(2t), 0)}{\sqrt{4\sin^2(2t) + 4\cos^2(2t)}} = \underline{(-\sin(2t), -\cos(2t), 0)}$

Unit Binormal

$\vec{B}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(2t) & -\sin(2t) & 0 \\ -\sin(2t) & -\cos(2t) & 0 \end{vmatrix} = \left[-\cos^2(2t) - \sin^2(2t)\right]\hat{k} = -\hat{k} = \underline{(0, 0, -1)}$

$\vec{r}'(t) \times \vec{r}''(t) = (0, 0, -8\cos^2(2t) - 8\sin^2(2t)) = (0, 0, -8)$
$\|\vec{r}'(t) \times \vec{r}''(t)\| = 8$

$\kappa = \dfrac{8}{2^3} = \underline{1}, \quad a_T = \underline{0}, \quad a_N = \dfrac{8}{2} = \underline{4}$

(b) $\vec{r}(t) = (\cos(t), \sin(t), t^2)$
$\vec{r}'(t) = (-\sin(t), \cos(t), 2t)$
$\vec{r}''(t) = (-\cos(t), -\sin(t), 2)$

$\vec{T}(t) = \dfrac{(-\sin(t), \cos(t), 2t)}{\sqrt{1 + 4t^2}} = \left(\dfrac{-\sin t}{\sqrt{1+4t^2}}, \dfrac{\cos t}{\sqrt{1+4t^2}}, \dfrac{2t}{\sqrt{1+4t^2}}\right)$

$\vec{N}(t) = \dfrac{(-(1+4t^2)\cos t + 4t\sin t, -(1+4t^2)\sin t - 4t\cos t, 2)}{\sqrt{16t^4 + 24t^2 + 5}}$

$\vec{B}(t) = \dfrac{(2\cos t + 2t\sin t, 2\sin t - 2t\cos t, 1)}{\sqrt{4t^2 + 5}}$

(A comment about the preceding nasty computations:

When doing beastly computations like the ones above it makes sense to check a bit as you go along. For example, I checked that $\vec{T} \cdot \vec{N} \equiv 0$ before I computed $\vec{B}$. To check my final answer, I checked that $\vec{T} \cdot \vec{B} \equiv 0$ & $\vec{N} \cdot \vec{B} \equiv 0$.) Moving on...

$\vec{r}'(t) \times \vec{r}''(t) = (2\cos t + 2t\sin t,\ 2\sin t - 2t\cos t,\ 1)$

$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{4t^2+5}$

$\vec{r}'(t) \cdot \vec{r}''(t) = 4t$

$\kappa = \dfrac{\sqrt{4t^2+5}}{(1+4t^2)^{3/2}}$ $\qquad a_T = \dfrac{4t}{\sqrt{1+4t^2}}$ $\qquad a_N = \sqrt{\dfrac{4t^2+5}{4t^2+1}}$

Phew...

2. #30 (a) – VI, (b) – V, (c) – I
(d) – IV (e) – II (f) – III

#55 – II – C #56 – IV – A
#57 – I – F #58 – III – E
#59 – VI – B #60 – V – D

3. $f(x,y)$ is continuous at the point (c,d) if $\lim_{(x,y)\to(c,d)} f(x,y)$ exists, (c,d) is in the domain of f, and $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$.

I do not mind if you abbreviate this defn. by saying,

"$f(x,y)$ is continuous at the point (c,d) if $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$."

4. (a) $f(x,y) = x^2 \sin(3xy^2 + 4x^2y)$

$f_x = 2x\sin(3xy^2 + 4x^2y) + x^2(3y^2 + 8xy)\cos(3xy^2 + 4x^2y)$

$f_y = x^2(6xy + 4x^2)\cos(3xy^2 + 4x^2y)$

$= 2x^3(3y + 2x)\cos(3xy^2 + 4x^2y)$

(b) $g(x,y) = x^3y^2e^{x^5y^4}$

$g_x = 3x^2y^2e^{x^5y^4} + x^3y^2 \cdot 5x^4y^4e^{x^5y^4}$

$= x^2y^2e^{x^5y^4}(3 + 5x^5y^4)$

$g_y = 2x^3ye^{x^5y^4} + x^3y^2 \cdot 4x^5y^3e^{x^5y^4}$

$= 2x^3ye^{x^5y^4}(1 + 2x^5y^4)$

(c) & (d) are pending

5. (a) $f(x,y) = x^2\sin(xy^2)$

$f_x = 2x\sin(xy^2) + x^2y^2\cos(xy^2)$

$f_y = 2x^3y\cos(xy^2)$

$f_{xx} = 2\sin(xy^2) + 2xy^2\cos(xy^2) + 2xy^2\cos(xy^2) - x^2y^4\sin(xy^2)$

$= (2 - x^2y^4)\sin(xy^2) + 4xy^2\cos(xy^2)$

$f_{xy} = f_{yx} = 6x^2y\cos(xy^2) - 2x^3y^3\sin(xy^2)$

$f_{yy} = 2x^3\cos(xy^2) - 4x^4y^2\sin(xy^2)$

(b) $g(x,y) = x^3y^2 - 4x^2y^3 + 3x^7y^9$

$g_x = 3x^2y^2 - 8xy^3 + 21x^6y^9$ $\quad g_y = 2x^3y - 12x^2y^2 + 27x^7y^8$

$g_{xx} = 6xy^2 - 8y^3 + 126x^5y^9$ $\quad g_{xy} = g_{yx} = 6x^2y - 24xy^2 + 189x^6y^8$

$g_{yy} = 2x^3 - 24x^2y + 216x^7y^7$

(c) & (d) are pending

6. (a) $f(x,y) = x^2y^3$ $\quad f(3,-1) = -9$

$f_x(x,y) = 2xy^3$ $\quad f_x(3,-1) = -6$

$f_y(x,y) = 3x^2y^2$ $\quad f_y(3,-1) = 27$

TP: $z = -9 + -6(x-3) + 27(y-(-1))$

$z = -9 - 6(x-3) + 27(y+1)$

LA: $L(x,y) = -9 - 6(x-3) + 27(y+1)$

(b) $g(x,y) = 3xy^2 - 4x^2y$ $\quad g(2,3) = 54 - 48 = 6$

$g_x(x,y) = 3y^2 - 8xy$ $\quad g_x(2,3) = 27 - 48 = -21$

$g_y(x,y) = 6xy - 4x^2$ $\quad g_y(2,3) = 36 - 16 = 20$

TP: $z = 6 - 21(x-2) + 20(y-3)$

LA: $L(x,y) = 6 - 21(x-2) + 20(y-3)$

7. (a) $F(x,y,z) = x^2 + y^4 + z^6 = 26$

$\nabla F(x,y,z) = (2x, 4y^3, 6z^5)$ $\quad \nabla F(3,-2,1) = (6,-32,6)$

TP: $(6,-32,6) \cdot (x-3, y+2, z-1) = 0$

Slightly better: $(3,-16,3) \cdot (x-3, y+2, z-1) = 0$

(b) $G(x,y,z) = xy + 2xz + 3yz = 14$

$\nabla G(x,y,z) = (y+2z, x+3z, 2x+3y)$

$\nabla G(-2,3,4) = (11, 10, 5)$

TP: $(11,10,5) \cdot (x+2, y-3, z-4) = 0$

8. (a) $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$

(b) $\frac{df}{dt} = \frac{\partial f}{\partial w}\cdot\frac{dw}{dt} + \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$

9. $f(x,y) = x + 3x^2y$

$\nabla f(x,y) = (1+6xy,\ 3x^2)$

(a) $\nabla f(-2,3) = (-35,\ 12)$ (b) $\nabla f(-1,4) = (-23, 3)$
(c) $\nabla f(5,0) = (1,\ 75)$

10. (a) In order... ① $\frac{(-35,12)}{\sqrt{1225+144}} = \frac{(-35,12)}{\sqrt{1369}}$

② $\frac{(-23,3)}{\sqrt{469+9}} = \frac{(-23,3)}{\sqrt{478}}$

③ $\frac{(1,75)}{\sqrt{5626}}$

(b) In order after observing $\frac{(3,4)}{\|(3,4)\|} = \left(\frac{3}{5}, \frac{4}{5}\right)$ ...

$(-35, 12)\cdot\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{-57}{5}$

$(-23, 3)\cdot\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{-57}{5}$

$(1, 75)\cdot\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{303}{5}$

11. (a) The pt. $(1,-2,5)$ is in the plane.

$(0,4,-2)$ is a vector from that pt. to the pt. $(1,2,3)$.

$(2,-3,4)$ is normal

$$\text{dist} = \left| (0,4,-2)\cdot\frac{(2,-3,4)}{\sqrt{4+9+16}} \right| = \frac{20}{\sqrt{29}}$$

(b) Dist^2 to pt. $= (x-1)^2 + (y-2)^2 + (z-3)^2$

Plane we're stuck in is

$$z = 5 + \tfrac{3}{4}(y+2) - \tfrac{1}{2}(x-1)$$

So we can minimize

$$F(x,y) = (x-1)^2 + (y-2)^2 + \left(2 + \tfrac{3}{4}(y+2) - \tfrac{1}{2}(x-1)\right)^2$$

$$\nabla F(x,y) = \Big(2(x-1) + 2\left(2+\tfrac{3}{4}(y+2) - \tfrac{1}{2}(x-1)\right)\left(-\tfrac{1}{2}\right),$$
$$2(y-2) + 2\left(2+\tfrac{3}{4}(y+2)-\tfrac{1}{2}(x-1)\right)\tfrac{3}{4}\Big) = (0,0)$$

x-comp. eqn. $2x - 2 - \left(2 + \tfrac{3}{4}(y+2) - \tfrac{1}{2}(x-1)\right) = 0$

$8x - 8 - 8 - 3(y+2) + 2(x-1) = 0$ (I mult thru by 4 to get rid of all frac's)

$$10x - 3y = 24$$

y-comp. eqn. $2y - 4 + \tfrac{3}{2}\left(2 + \tfrac{3}{4}(y+2) - \tfrac{1}{2}(x-1)\right) = 0$

$16y - 32 + 24 + 9(y+2) - 6(x-1) = 0$ (Here I mult'd by 8.)

$$6x - 25y = 16$$

$30x - 9y = 72$
$-30x + 125y = -80$
$116y = -8$
$29y = -2$ $\quad y = -\frac{2}{29} \Rightarrow 10x = 24 - \frac{6}{29}$

$290x = 696 - 6 = 690$

$x = \frac{69}{29}$

$$F\left(\tfrac{69}{29}, -\tfrac{2}{29}\right) = \left(\tfrac{40}{29}\right)^2 + \left(\tfrac{-60}{29}\right)^2 + \left(2 + \tfrac{3}{4}\left(\tfrac{56}{29}\right) - \tfrac{1}{2}\left(\tfrac{40}{29}\right)\right)^2$$

$$= \frac{1600}{29^2} + \frac{3600}{29^2} + \left(\frac{58}{29} + \frac{42}{29} - \frac{20}{29}\right)^2 = \frac{5200 + 6400}{29^2} = \frac{11600}{29^2}$$

$$\begin{array}{r} 400 \\ 29\overline{)11600} \\ \underline{116} \end{array}$$

$= \frac{400}{29}$ Which is the square of the dist. to the pt. So the answer is $20/\sqrt{29}$. (YAY!)

(c) Minimize $f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$
Subject to: $g(x,y,z) = 2(x-1) - 3(y+2) + 4(z-5) = 0$

$\nabla f = \left(2(x-1),\ 2(y-2),\ 2(z-3)\right)$

$\nabla g = (\ 2,\ -3,\ 4\ )$

$\nabla f = \lambda \nabla g$ so...

$$2(x-1) = 2\lambda$$
$$2(y-2) = -3\lambda$$
$$2(z-3) = 4\lambda$$
$$2(x-1) - 3(y+2) + 4(z-5) = 0$$

$$\frac{\lambda}{2} = \frac{(x-1)}{2} = \frac{(2-y)}{3} = \frac{(z-3)}{4} \Rightarrow 2(x-1) = z-3$$

$$4(2-y) = 3(z-3)$$
$$(z-3) - 3(y+2) + 4(z-5) = 0$$

$$4y + 3z = 17 \qquad 12y + 9z = 51$$
$$-3y + 5z = 29 \qquad -12y + 20z = 116$$
$$29z = 167$$
$$z = \frac{167}{29} \qquad z-3 = \frac{80}{29}$$

$$2-y = \frac{60}{29} \Rightarrow y = -\frac{2}{29} \qquad x-1 = \frac{40}{29} \Rightarrow x = \frac{69}{29}$$

$$f\left(\frac{69}{29}, -\frac{2}{29}, \frac{167}{29}\right) = \frac{1600 + 3600 + 6400}{29^2}$$

and now the arithmetic is the same as the end of part (b).

12. $f(x,y) = 16x^3 + 2xy^2 + 20x^2 + y^2$

$$f_x = 48x^2 + 2y^2 + 40x = 2(24x^2 + 20x + y^2)$$
$$f_y = 4xy + 2y = 2y(2x+1)$$
$$f_{xx} = 96x + 40 \qquad f_{xy} = 4y \qquad f_{yy} = 4x + 2$$

Setting $f_y = 0 \Rightarrow$ either $y=0$ or $x = -\frac{1}{2}$.

If $y=0$, then $0 = f_x = 48x^2 + 40x = 8x(6x+5)$

so $x = 0$ or $-\frac{5}{6}$

If $x = -\frac{1}{2}$ then $0 = f_x = 12 + 2y^2 - 20 \Rightarrow 4 = y^2$
$\Rightarrow y = \pm 2$

We have the CPs: $(0,0), (-\frac{5}{6}, 0), (-\frac{1}{2}, -2), (-\frac{1}{2}, 2)$

$D(0,0) > 0$, $f_{xx}(0,0) > 0$ so $(0,0)$ is a min.

$D(-\frac{5}{6}, 0) = \left(96(-\frac{5}{6}) + 40\right)\left(4(-\frac{5}{6}) + 2\right) = (-40)(-\frac{8}{6}) > 0$

$f_{xx}(-\frac{5}{6}, 0) = -40 < 0$ so $(-\frac{5}{6}, 0)$ is a min.

$$\left.\begin{array}{l} D(-\frac{1}{2}, -2) = (-8)(0) - 64 = -64 < 0 \\ D(-\frac{1}{2}, 2) = (-8)(0) - 64 = -64 < 0 \end{array}\right\} \text{Both } (-\tfrac{1}{2}, -2) \text{ \& } (-\tfrac{1}{2}, +2) \text{ are saddles.}$$

13. $F(x,y) = 3x - x^3 - 2y^2 + y^4$

$F_x = 3 - 3x^2 = 0 \Rightarrow x = \pm 1$

$F_y = -4y + 4y^3 = 4y(y^2 - 1) \Rightarrow y = 0, \pm 1$

6 CPs: $(1,0), (1, \pm 1), (-1, 0), (-1, \pm 1)$

$F(1,0) = 2$, $F(1, \pm 1) = 1$, $F(-1, 0) = -2$, $F(-1, \pm 1) = -3$

$f(x) := F(x, -2) = \underset{\uparrow \text{Lucky us!}}{=} F(x, 2) = 3x - x^3 - 8 + 16 = 8 + 3x - x^3$

Do max/min of $f(x)$ on $-2 \le x \le 2$: $f'(x) = 3 - 3x^2 = 0$
so $x = \pm 1$ are CPs

$f(-2) = 8 - 6 + 8 = 10$, $f(2) = 6$, $f(-1) = 6$, $f(1) = 10$

$g(y) := F(-2, y) = 2 - 2y^2 + y^4$

max/min on $-2 \le y \le 2$: $g'(y) = -4y + 4y^3 = 4y(y^2 - 1)$ $y = 0, \pm 1$

$g(-2) = f(-2) = 10$, $g(2) = f(-2) = 10$ $g(0) = 2$, $g(\pm 1) = 1$

$h(y) := F(2,y) = -2-2y^2+y^4$ $\quad h'(y)=g'(y)$ so $y=0,\pm 1$ are the CPs

$h(-2) = f(2) = 6, \; h(2) = f(2) = 6$

$h(0) = -2, \; h(\pm 1) = -3$

Ginormous ties for mins & max's

$F(-1,\pm 1) = F(2,\pm 1) = -3$ for minimums

$F(-2,\pm 2) = F(1,\pm 2) = 10$ for maximums

14. (a) $\nabla F = (e^{2y+3z}, 2xe^{2y+3z}, 3xe^{2y+3z})$

$\nabla G = (4x^3-y+2z, 4y^3-x-3z, 4z^3+2x-3y)$

$\nabla F = \lambda \nabla G$

4 unknowns: (λ, x, y, z)

4 eqns: $\begin{cases} e^{2y+3z} = \lambda(4x^3-y+2z) \\ 2xe^{2y+3z} = \lambda(4y^3-x-3z) \\ 3xe^{2y+3z} = \lambda(4z^3+2x-3y) \\ x^4+y^4+z^4-xy+2xz-3yz=40 \end{cases}$

(b) $\nabla F = (2xy^3+3x^2y^2, 3x^2y^2+2x^3y, -1)$

$\nabla G = (2x, 4(2y+1), 6(3z-1))$

$\nabla F = \lambda \nabla G$

4 unknowns: (λ, x, y, z)

4 eqns: $\begin{cases} 2xy^3+3x^2y^2 = \lambda 2x \\ 3x^2y^2+2x^3y = \lambda 4(2y+1) \\ -1 = \lambda 6(3z-1) \\ x^2+(2y+1)^2+(3z-1)^2 = 7^2 \end{cases}$

(c) $\nabla F = (ye^{yz}, xe^{yz}+xyze^{yz}, xy^2e^{yz})$

$\nabla G = (2x, 2y, 2z)$

$\nabla H = (y+2z, x-z, 2x-y)$

$\nabla F = \lambda \nabla G + \mu \nabla H$

5 unknowns: (λ, μ, x, y, z)

5 eqns: $\begin{cases} ye^{yz} = \lambda 2x + \mu(y+2z) \\ xe^{yz}(1+yz) = \lambda 2y + \mu(x-z) \\ xy^2e^{yz} = \lambda 2z + \mu(2x-y) \\ x^2+y^2+z^2 = 5^2 \\ xy+2xz-yz=10 \end{cases}$

15. max/min $f(x,y) = x^2-2x+y^2-4y$ in $(x-4)^2+(y+3)^2 \le 8^2$

Part I: deal with $(x-4)^2+(y+3)^2 < 8^2$ by setting $\nabla f = 0$

$\nabla f = (2x-2, 2y-4) = (0,0) \Rightarrow x=1, y=2. \quad f(1,2) = -1-4 = -5$

Part II: deal with $g(x,y) = (x-4)^2 + (y+3)^2 = 8^2$
by setting $\nabla f = \lambda \nabla g = \lambda(2(x-4), 2(y+3))$

$$2x - 2 = 2\lambda(x-4) \Rightarrow x - 1 = \lambda(x-4)$$
$$2y - 4 = 2\lambda(y+3) \Rightarrow y - 2 = \lambda(y+3)$$

$$\frac{x-1}{x-4} = \frac{y-2}{y+3} \Rightarrow xy + 3x - y - 3 = xy - 4y - 2x + 8$$
$$5x + 3y = 11$$
$$(x-4)^2 + (y+3)^2 = 8^2$$

$$x = \frac{11-3y}{5} \Rightarrow \left(\frac{11-3y-20}{5}\right)^2 + (y+3)^2 = 8^2$$
$$\left(\frac{-3y-9}{5}\right)^2 + (y+3)^2 = 8^2$$
$$\frac{9}{25}(y+3)^2 + (y+3)^2 = 8^2$$
$$\frac{34}{25}(y+3)^2 = 8^2 \qquad (y+3)^2 = \frac{40^2}{34}$$
$$y+3 = \frac{\pm 40}{\sqrt{34}}$$

$$y = -3 \pm \frac{40}{\sqrt{34}}$$

$$y = \frac{11-5x}{3} \Rightarrow (x-4)^2 + \left(\frac{20-5x}{3}\right)^2 = 8^2$$
$$(x-4)^2 + \left(\frac{5}{3}\right)^2 (x-4)^2 = 8^2$$
$$\frac{34}{9}(x-4)^2 = 8^2 \qquad (x-4)^2 = \frac{24^2}{34} \qquad x-4 = \frac{\pm 24}{\sqrt{34}}$$

$$x = 4 \pm \frac{24}{\sqrt{34}}$$

Because of the relation $5x + 3y = 11$ we will get $\left(4 - \frac{24}{\sqrt{34}}, -3 + \frac{40}{\sqrt{34}}\right)$ & $\left(4 + \frac{24}{\sqrt{34}}, -3 - \frac{40}{\sqrt{34}}\right)$
as the points to plug in to f to compare with $f(1,2) = -5$.