1 (a) $\int_0^3 \int_0^4 (xy^2 - x^2y)\,dx\,dy = \int_0^3 \left(\frac{x^2}{2}y^2 - \frac{x^3}{3}y\right)\Big|_0^4 \,dy$

$= \int_0^3 \left(8y^2 - \frac{64}{3}y\right) dy$

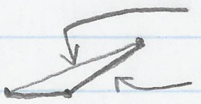
$= \left(8\frac{y^3}{3} - \frac{64}{3}\frac{y^2}{2}\right)\Big|_0^3$

$= 72 - 32\cdot 3 = 72 - 96 = -24$

(b) $\int_0^2 \int_0^5 \int_{-1}^0 xyz\;dz\;dy\;dx = \int_0^2 x\,dx \int_0^5 y\,dy \int_{-1}^0 z\,dz$

$= \frac{2^2}{2}\cdot\frac{5^2}{2}\cdot\left(-\left(\frac{(-1)^2}{2}\right)\right) = -\frac{25}{2}$

2 (a)



$y = \frac{x}{2} \Leftrightarrow x = 2y$

$y = x - 1 \Leftrightarrow x = y + 1$

$\iint_T 12xy\,dA = \int_{y=0}^{1} \int_{x=2y}^{y+1} 12xy\,dx\,dy$ (If I want to do this as "dy dx", then I must chop it at $x = 1$.)

$= \int_{y=0}^{1} 6x^2y\Big|_{2y}^{(y+1)} dy$

$= \int_0^1 \left[6(y+1)^2y - 6(2y)^2y\right] dy$

$= \int_0^1 \left[6y^3 + 12y^2 + 6y - 24y^3\right] dy$

$= \int_0^1 (6y + 12y^2 - 18y^3)\,dy = \left(3y^2 + 4y^3 - \frac{9}{2}y^4\right)\Big|_0^1$

$= 7 - \frac{9}{2} = \frac{5}{2}$

$\vec{V}_1 := (0,0,0)$ $\quad \vec{V}_3 = (0,3,0)$

$\vec{V}_2 := (2,0,0)$ $\quad \vec{V}_4 = (0,0,6)$

(b) The vertices $\vec{V}_1, \vec{V}_2, \vec{V}_3$ all have z-coord $= 0$, so they belong to the plane $z=0$ since $\vec{V}_4$ has positive z-coord. we have the inequality $z \geq 0$. Consideration of the other faces which include $\vec{V}_1$ lead us to $x \geq 0$ & $y \geq 0$. The hard part is finding the face containing $\vec{V}_2, \vec{V}_3,$ & $\vec{V}_4$.

For that face you need to remember the stuff from the first unit about finding a plane from three points. In any case the plane containing $\vec{V}_2, \vec{V}_3,$ & $\vec{V}_4$ is given by

$$3x + 2y + z = 6$$

(You may need to do a computation to get this eqn.) Since we want $\vec{V}_1$ to belong to the set we have

$$3x + 2y + z \leq 6$$

I am hoping that everyone can now get to here or something similar

$$\int_{x=0}^{2} \int_{y=0}^{☺} \int_{z=0}^{6-3x-2y} x \, dz \, dy \, dx$$

So... How to find "☺"?

1^{ST} Solve $3x + 2y + z = 6$ for y

$$\Rightarrow \quad y = \frac{6-3x-z}{2}$$

2^{ND} Since it is the upper bound of integration we maximize it over

all z while x is held fixed.
In this case z=0 gives us the maximum so we get

$$\int_{x=0}^{2} \int_{y=0}^{\frac{6-3x}{2}} \int_{z=0}^{6-3x-2y} x \, dz\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{\frac{6-3x}{2}} 6x-3x^2-2yx \, dy\, dx$$

$$= \int_{x=0}^{2} \left(6xy - 3x^2y - xy^2\right)\Big|_{y=0}^{\frac{6-3x}{2}} dx$$

$$= \int_{x=0}^{2} \left[x(6-3x)\left(\frac{6-3x}{2}\right) - x\left(\frac{6-3x}{2}\right)\left(\frac{6-3x}{2}\right)\right] dx$$

$$=\left(\frac{1}{2}-\frac{1}{4}\right)\int_{x=0}^{2} x(6-3x)^2\, dx = \frac{1}{4}\int_{x=0}^{2} \left(9x^3 - 36x^2 + 36x\right) dx$$

$$= \frac{1}{4}\left(9\,\frac{2^4}{4} - 12\cdot 2^3 + 18\cdot 2^2\right) = \frac{1}{4}\left(9\cdot 2^2 - 24\cdot 2^2 + 18\cdot 2^2\right)$$

$$= 3.$$

(c) 1^{ST} Quad $\Rightarrow$ $x \geq 0, y \geq 0$ between circles $r=1$ & $r=2$. Using polar coordinates

$$\iint_R x \, dA = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=1}^{2} (r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_{r=1}^{2} r^2 dr = \sin\theta\Big|_0^{\frac{\pi}{2}} \cdot \frac{r^3}{3}\Big|_1^2 = 1\cdot\frac{7}{3} = \frac{7}{3}.$$

(d) $\iiint_R z\, dV = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{\rho=0}^{4} \rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$$= \frac{\pi}{2} \int_{\phi=0}^{\frac{\pi}{2}} \cos\phi \sin\phi\, d\phi \cdot \int_{\rho=0}^{4} \rho^3\, d\rho$$

$$= \frac{\pi}{2} \cdot \frac{4^4}{4} \cdot \int_{u=0}^{1} u\, du = 16\pi$$

$u = \sin\phi$

3 (a) Let $\vec{V}_1 = \vec{0}$, $\vec{V}_2 = (2,0,0)$, $\vec{V}_3 = (2,4,0)$, $\vec{V}_4 = (2,4,1)$

$\vec{V}_1, \vec{V}_2, \vec{V}_3$ belong to $z=0$. By using $\vec{V}_4$ we get $z \geq 0$.

$\vec{V}_2, \vec{V}_3, \vec{V}_4$ belong to $x=2$. By using $\vec{V}_1$ we get $x \leq 2$.

$\vec{V}_1, \vec{V}_2, \vec{V}_4$ belong to $y=4z$. Using $\vec{V}_3 \Rightarrow y \geq 4z$.

$\vec{V}_1, \vec{V}_3, \vec{V}_4$ belong to $y=2x$. Using $\vec{V}_2 \Rightarrow y \leq 2x$.

$z \geq 0$, $x \leq 2$, $4z \leq y \leq 2x \Rightarrow 0 \leq 4z \leq y \leq 2x \leq 4$

$0 \leq z \leq \frac{y}{4} \leq \frac{x}{2} \leq 1$

$0 \leq 2z \leq \frac{y}{2} \leq x \leq 2$.

So $\iiint_T x\, dV = \int_{z=0}^{1} \int_{y=4z}^{4} \int_{x=\frac{y}{2}}^{2} x\, dx\, dy\, dz$

$$= \int_{z=0}^{1} \int_{x=2z}^{2} \int_{y=4z}^{2x} x\, dy\, dx\, dz = \int_{y=0}^{4} \int_{z=0}^{\frac{y}{4}} \int_{x=\frac{y}{2}}^{2} x\, dx\, dz\, dy$$

$$= \int_{y=0}^{4} \int_{x=\frac{y}{2}}^{2} \int_{z=0}^{\frac{y}{4}} x\, dz\, dx\, dy = \int_{x=0}^{2} \int_{y=0}^{2x} \int_{z=0}^{\frac{y}{4}} x\, dz\, dy\, dx = \int_{x=0}^{2} \int_{z=0}^{\frac{x}{2}} \int_{y=4z}^{2x} x\, dy\, dz\, dx$$

(All of these are correct.)

(b) $6 \le \rho \le 12$, 1^{ST} Octant $\Rightarrow$ $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$

$z \le \sqrt{x^2+y^2}$ further restricts ϕ to $\frac{\pi}{4} \le \phi$

$$\iiint_R x\,dV = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\rho=6}^{12} \rho \sin\phi \cos\theta \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

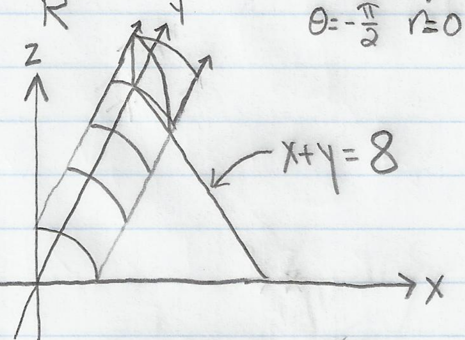
$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\rho=6}^{12} \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta$$

(c) (Cylindrical Coords are easiest here.)

$x \ge 0$, $x^2+y^2=r^2 \le 16$, $-x \le z \le x^2+y^2 = r^2$

$$\iiint_R z\,dV = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{4} \int_{z=-r\cos\theta}^{r^2} z \, dz \, r \, dr \, d\theta$$

(d)



$x, y, z \ge 0$, $x^2+z^2 \le 4$
$x+y \le 8$

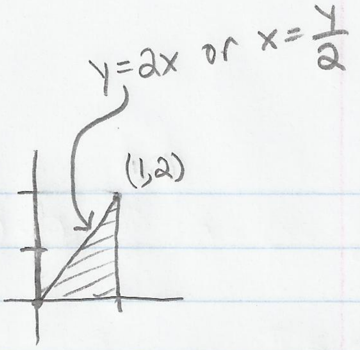
The "Shadow" (Projection) in the xz plane is just a quarter circle.

$$\int_{z=0}^{2} \int_{x=0}^{\sqrt{4-z^2}} \int_{y=0}^{8-x} (x+y+z)\,dy\,dx\,dz$$

or

$$\int_{x=0}^{2} \int_{z=0}^{\sqrt{4-x^2}} \int_{y=0}^{8-x} (x+y+z)\,dy\,dz\,dx$$

4. (a) $\displaystyle\int_{y=0}^{2}\int_{x=\frac{y}{2}}^{1} e^{x^2}\,dx\,dy = I_1$



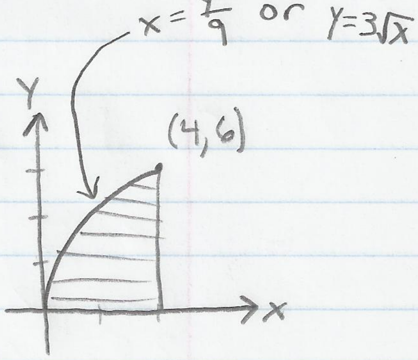
$$0 \le y \le 2$$
$$0 \le \frac{y}{2} \le x \le 1$$
$$0 \le y \le 2x \le 2$$

$$I_1 = \int_{x=0}^{1}\int_{y=0}^{2x} e^{x^2}\,dy\,dx$$

$$= \int_{x=0}^{1} 2xe^{x^2}\,dx = e^{x^2}\Big|_0^1 = e^1 - e^0$$

$$= e - 1.$$

(b) $\displaystyle\int_{y=0}^{6}\int_{x=\frac{y^2}{9}}^{4} e^{x^{(3/2)}}\,dx\,dy = I_2$



$$0 \le y \le 6$$
$$0 \le \frac{y^2}{9} \le x \le 4$$
$$0 \le y^2 \le 9x \le 36$$
$$0 \le y \le 3\sqrt{x} \le 6$$

$$I_2 = \int_{x=0}^{4}\int_{y=0}^{3\sqrt{x}} e^{x^{(3/2)}}\,dy\,dx$$

$$= \int_{x=0}^{4} 3x^{\frac{1}{2}} e^{x^{\frac{3}{2}}}\,dx$$

$$\underset{\substack{\uparrow \\ u = x^{\frac{3}{2}}}}{=} \int_0^8 e^u \cdot 2\,du = 2(e^8 - 1)$$

$$du = \frac{3}{2}x^{\frac{1}{2}}\,dx$$

5. (a) $0 \le z \le 2$, $0 \le y \le \frac{6-3z}{2}$, $0 \le x \le 6-2y-3z$

So... $x+2y+3z \le 6$ which together with $0 \le x$, $0 \le y$, $0 \le z$ already gives us everything we need.

$$\int_{z=0}^{2} \int_{y=0}^{\frac{6-3z}{2}} \int_{x=0}^{6-2y-3z} \rho \, dx \, dy \, dz$$

$$= \int_{z=0}^{2} \int_{x=0}^{6-3z} \int_{y=0}^{\frac{6-x-3z}{2}} \rho \, dy \, dx \, dz$$

$$= \int_{y=0}^{3} \int_{x=0}^{6-2y} \int_{z=0}^{\frac{6-x-2y}{3}} \rho \, dz \, dx \, dy$$

$$= \int_{y=0}^{3} \int_{z=0}^{\frac{6-2y}{3}} \int_{x=0}^{6-2y-3z} \rho \, dx \, dz \, dy$$

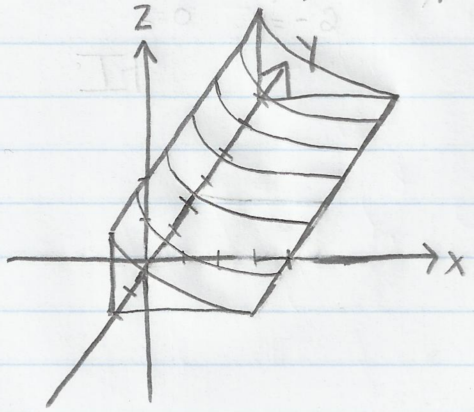
$$= \int_{x=0}^{6} \int_{z=0}^{\frac{6-x}{3}} \int_{y=0}^{\frac{6-x-3z}{2}} \rho \, dy \, dz \, dx$$

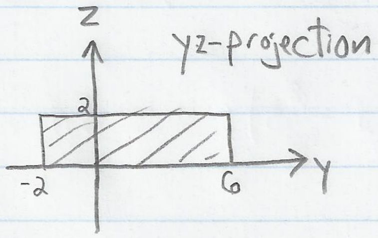
$$= \int_{x=0}^{6} \int_{y=0}^{\frac{6-x}{2}} \int_{z=0}^{\frac{6-x-2y}{3}} \rho \, dz \, dy \, dx$$

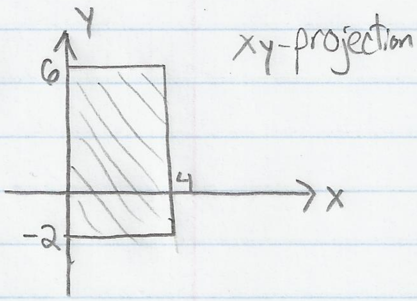
(b) $0 \le z \le 2$, $-2 \le y \le 6$, $0 \le x \le (z-2)^2 = (2-z)^2$

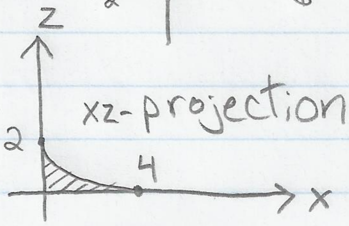
Notice that when $z=2$, $(z-2)^2=0$

" $z=0$, $(z-2)^2=4$









$$\int_{z=0}^{2}\int_{y=-2}^{6}\int_{x=0}^{(z-2)^2} \rho \, dx\, dy\, dz$$

$$= \int_{z=0}^{2}\int_{x=0}^{(z-2)^2}\int_{y=-2}^{6} \rho \, dy\, dx\, dz$$

$$= \int_{y=-2}^{6}\int_{z=0}^{2}\int_{x=0}^{(z-2)^2} \rho \, dx\, dz\, dy$$

$$= \int_{y=-2}^{6}\int_{x=0}^{4}\int_{z=0}^{2-\sqrt{x}} \rho \, dz\, dx\, dy$$

$$= \int_{x=0}^{4}\int_{z=0}^{2-\sqrt{x}}\int_{y=-2}^{6} \rho \, dy\, dz\, dx$$

$$= \int_{x=0}^{4}\int_{y=-2}^{6}\int_{z=0}^{2-\sqrt{x}} \rho \, dz\, dy\, dx$$

I have accidentally done something evil. $z-2$ is negative in the domain in question so before taking the square root of

$$x \le (z-2)^2$$

it is wise to rewrite it as

$$x \le (2-z)^2.$$

6.(a) $\nabla \cdot \vec{F} = 3$, $\nabla \times \vec{F} = (-5,-5,-5)$

(b) $\nabla \cdot \vec{G} = 0$, $\nabla \times \vec{G} = (-2y-2z, -2z-2x, -2x-2y)$

(c) $\nabla \cdot \vec{H} = \sin y + \cos x + 2z$, $\nabla \times \vec{H} = (0, 0, -y\sin x - x\cos y)$

7.(a) You can take the curl of $\vec{F}$ extended to be zero in the third coordinate, or you can use Theorem 6 on p. 1050. Anyway...

$$\text{Curl}\left(2x\cos(xy) + x^2y\sin(xy), -x^3\cos(xy), 0\right)$$

$$= \left(0,0,-3x^2\cos(xy) + x^3y\sin(xy) - \left[-2x^2\sin(xy) + x^2\sin(xy) + x^3y\cos(xy)\right]\right)$$

$$\neq \vec{0}$$

(b) $\text{Curl}\left(2x\cos(xy)-x^2y\sin(xy),\ x^3\cos(xy),0\right)$

$$=\left(0,0,\ 3x^2\cos(xy)-x^3y\sin(xy)-\left[-2x^2\sin(xy)-x^2\sin(xy)-x^3y\cos(xy)\right]\right)$$
$$\neq \vec{0}$$

(c) $\text{Curl}\left(2x\cos(xy)-x^2y\sin(xy),-x^3\cos(xy),0\right)$

$$=\left(0,0,-3x^2\cos(xy)+x^3y\sin(xy)-\left[-2x^2\sin(xy)-x^2\sin(x,y)-x^3y\cos(xy)\right]\right)$$
$$\neq \vec{0}$$

Ack! I messed up. Somehow, I flipped around my cosines and sines when creating part of the problem.

Here is the vector field I intended:

$$\vec{H}=\left(2x\cos(xy)-x^2y\sin(xy),\ -x^3\sin(xy)\right)$$

and, yeah, its curl is $\vec{0}$. I will solve the rest of this problem on a supplemental handout.

(d) $\nabla\times\vec{K}=(x-x,\ y-y,\ z-z)=\vec{0}$. Yay!

So $\vec{K}=(yz+1,\ xz+2y,\ xy+3z^2)$ is conservative and must equal the gradient of a scalar field Θ. So, if $\nabla\Theta=\vec{K}$, then:

$$\Theta_x=yz+1 \Rightarrow \Theta=\int(yz+1)dx=xyz+x+P(y,z)$$

$$\Theta_y=xz+2y \Rightarrow \Theta=\int(xz+2y)dy=xyz+y^2+Q(x,z)$$

$$\Theta_z=xy+3z^2 \Rightarrow \Theta=\int(xy+3z^2)dz=xyz+z^3+R(x,y)$$

So letting $H(x,y,z) = xyz + x + y^2 + z^3 + \text{Const},$ we can make it match up each of the three previous lines. In this case we have

$$P(y,z) = y^2 + z^3 + \text{Const.}$$
$$Q(x,z) = x + z^3 + \text{Const.}$$
$$R(x,y) = x + y^2 + \text{Const.}$$

Thus, $\boxed{H(x,y,z) = xyz + x + y^2 + z^3 + \text{Const.}}$

is the answer.

Another Way:

Starting exactly as above by taking an anti-derivative in the x-variable we get to:

$$H(x,y,z) = xyz + x + P(y,z) \qquad (*)$$

However, now, instead of taking an anti-derivative of $xz + 2y$ w.r.t. y and playing a "match up game" like we did above we take a derivative of $(*)$ w.r.t y to try to find $P(y,z)$ more explicitly:

$$\frac{\partial P}{\partial y}(y,z) + xz + 0 = H_y = xz + 2y \leftarrow \text{(the second component of } \vec{K}\text{)}$$

which gives us $\frac{\partial P}{\partial y}(y,z) = 2y$

Thus $P(y,z) = \int 2y\,dy = y^2 + \tilde{P}(z)$

and $(*)$ becomes:

$$H(x,y,z) = xyz + x + y^2 + \tilde{P}(z) \qquad \widetilde{(*)}$$

Now again we take a deriv. of $\tilde{\circledast}$, but w.r.t. z and setting it equal to the third component of $\vec{K}$, we have

$$\tilde{P}'(z) + xy + 0 + 0 = H_z = xy + 3z^2$$

which gives us $\tilde{P}'(z) = 3z^2 \Rightarrow \tilde{P}(z) = z^3 + \text{Const.}$

and $\tilde{\circledast}$ becomes

$$\boxed{H(x,y,z) = xyz + x + y^2 + z^3 + \text{Const.}}$$

8. ① $\vec{F}$ incompressible $\Leftrightarrow$ div $\vec{F} = 0$
② $\vec{F}$ irrotational $\Leftrightarrow$ curl $\vec{F} = \vec{0}$
③ $\vec{F}$ is conservative $\Leftrightarrow$ there exists an f s.t. $\vec{F} = \nabla f$.
④ If $\vec{F} = \nabla f$, then f is a potential fct.

Everything above is a matter of definition, and ③ & ④ are obviously connected to each other. What is more interesting, however, is that for any simply connected domain

$$\text{Curl } \vec{F} = \vec{0} \Leftrightarrow \vec{F} \text{ is conservative.}$$