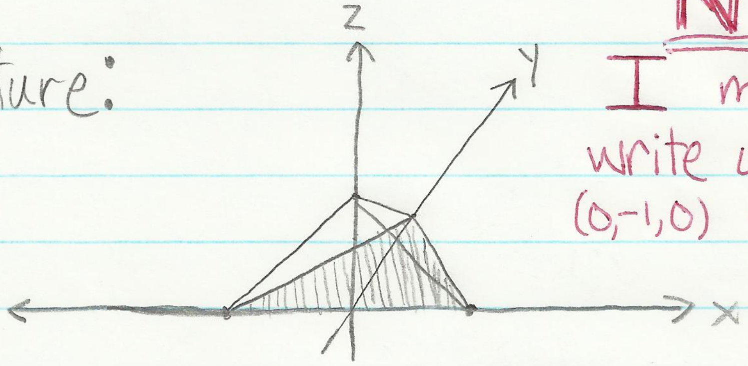
1. $$\iiint_{\Omega} \operatorname{div} \vec{F}\, dV = \iint_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS$$ ($\vec{n}$ is outward unit normal.)

$$\iint_{D} \operatorname{div} \vec{F}\, dV = \int_{\partial D} \vec{F} \cdot \vec{n}\, ds$$ "

2. $$\int_{C} \nabla f \cdot d\vec{r} = f(\text{finish}) - f(\text{start})$$

3. Picture:



NOTE: I messed up in the write up. I replaced (0,-1,0) with (-1,0,0) thereby interchanging the roles of x & y.

Answer: dx dy dz & dx dz dy work best.
(In the original problem dy dx dz & dy dz dx work best.)

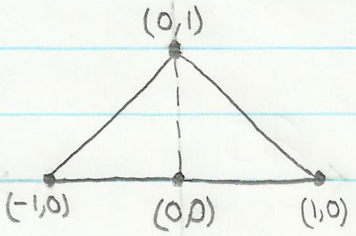
With any other order you must chop the tetrahedral along the yz-plane. (xz)

To see this fact imagine integrating w.r.t. z first. In this case one would start at a point in the triangle with vertices (1,0,0), (0,1,0), & (-1,0,0) (which I shaded) and integrating in the z-direction one would stop when one reached the triangle w/ vert's (-1,0,0), (0,0,1), (0,1,0) (which is the case if the initial point had negative x-coordinate) OR when one reached the triangle w/ vert!s $(1,0,0)$, $(0,0,1)$, $(0,1,0)$ (which is the case if the initial point had a positive x-coordinate).

The Key word in the preceding sentence is "OR." Because the simple fct. which determines the "roof" changes when the initial pt. crosses the y-axis integrating dz first requires that we chop the tetrahedral.

Another way to see that dz is a bad way to start is to observe that when we project to the xy-plane we still have four distinct vertices. Namely $(-1,0)$, $(0,0)$, $(1,0)$, $(0,1)$.

Bird's eye view:



I labeled the projection of the "crease" with a dotted line.

The argument for why you don't want to integrate w.r.t. y (x) first is similar.

On the other hand, if you start by integrating w.r.t. x (y), then the left-hand starting point will ALWAYS be on the triangle with vertices $(-1,0,0)$, $(0,0,1)$, $(0,1,0)$, and the right-hand ending point will ALWAYS be on the triangle with vertices $(1,0,0)$, $(0,0,1)$, $(0,1,0)$.

Another way to see that dx is a good way to start is to observe that $(1,0,0)$ & $(-1,0,0)$ BOTH project to $(0,0)$ in the yz-plane.

Once you have integrated w.r.t. x you must integrate over ~~the~~ the triangle in the yz-plane w/ vert!s $(0,0)$, $(1,0)$, $(0,1)$. One can obviously now integrate $dy\,dz$ or $dz\,dy$.

Apologies for messing up the vertices.

4. The best way to start is to observe that $\nabla \times \vec{G} = 0$. Letting $\vec{G} = \nabla h$, we have

$$h_x = yz \quad \Rightarrow h(x,y,z) = xyz + P(y,z)$$
$$h_y = xz + 3y^2 \quad \Rightarrow h(x,y,z) = xyz + y^3 + Q(x,z)$$
$$h_z = xy + 2z \quad \Rightarrow h(x,y,z) = xyz + z^2 + R(x,y)$$

Matching up terms we get

$$h(x,y,z) = xyz + y^3 + z^2 + \text{Constant}$$

We may as well take the constant to be zero & get $h(x,y,z) = xyz + y^3 + z^2$.

Knowing this makes a few problems trivial.

4a) $C := \{\vec{r}(t) = (5\cos t, 5\sin t, 0),\ 0 \le t \le 2\pi\}$

or $= \{$ " " " " " $, -\pi \le t \le \pi\}$

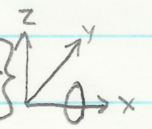
or others...

$$\int_C \vec{G} \cdot d\vec{r} = \int_C \nabla h \cdot d\vec{r} = f(\text{something}) - f(\text{same thing}) = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-\pi}^{\pi} (5\cos t, 10\cos t, 25\sin^2 t) \cdot (-5\sin t, 5\cos t, 0)\,dt$$

$$= \int_{-\pi}^{\pi} (50\cos^2 t - 25\sin t\cos t)\,dt = 50\pi$$

b) $C := \{\vec{r}(t) = (4, 3\sin t, 4\cos t), -\pi \le t \le \pi\}$



Again (& for the same reasons) $\int_C \vec{G} \cdot d\vec{r} = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-\pi}^{\pi} (4, 8 + 12\sin t\cos t, 9\sin^2 t - 16\cos^2 t) \cdot (0, 3\cos t, -4\sin t)\,dt$$

$$= \int_{-\pi}^{\pi} \left[24\cos t + 36\sin t\cos^2 t - 36\sin^3 t + 64\cos^2 t\sin t\right] dt$$

$$= 0.$$

c) $C := \{\vec{r}(t) = (t, t^2+1, 3),\ -2 \le t \le 4\}$

$$\int_C \vec{G} \cdot d\vec{r} = h(4,17,3) - h(-2,5,3)$$

$$= (4 \cdot 17 \cdot 3 + 17^3 + 3^2) - (-2 \cdot 5 \cdot 3 + 5^3 + 3^2)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-2}^{4} (t, 2t + 3(t^2+1), (t^2+1)^2 - 3^2) \cdot (1, 2t, 0)\,dt$$

$$= \int_{-2}^{4} (t + 4t^2 + 6t^3 + 6t)\,dt = \int_{-2}^{4} (6t^3 + 4t^2 + 7t)\,dt$$

d) $C := \{\vec{r}(t) = t(0,1,0) + (1-t)(1,0,1),\quad 0 \le t \le 1\}$
$= \{\vec{r}(t) = (1-t, t, 1-t),\quad 0 \le t \le 1\}$

$$\int_C f(x,y,z)\,dx + g(x,y,z)\,dy$$
$$= \int_0^1 \left(1 + (1-t)^2 e^{1-t} + t^4 e^{2t} + (1-t)^6 e^{3(1-t)}\right)(-1)\,dt$$
$$+ \int_0^1 (1-t)\cdot t\cdot(1-t)\cdot(1)\,dt$$

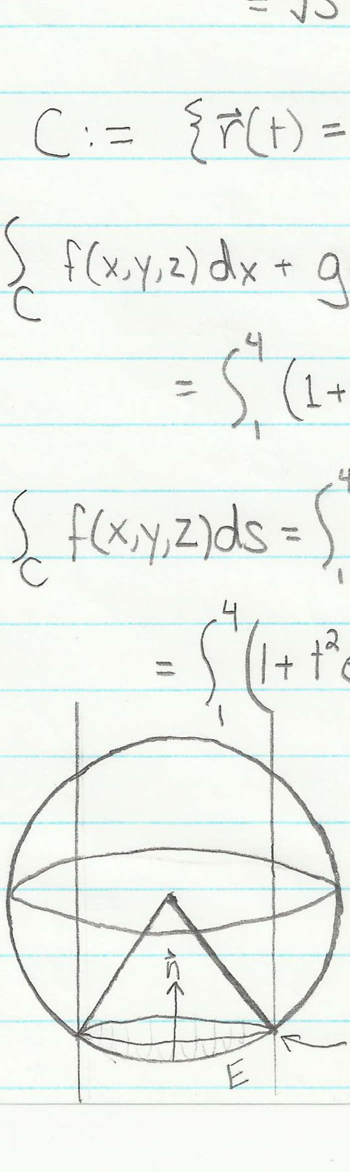
$$\int_C f(x,y,z)\,ds = \int_0^1 \left(1 + (1-t)^2 e^{1-t} + t^4 e^{2t} + (1-t)^6 e^{3(1-t)}\right)\cdot \|(-1,1,-1)\|\,dt$$
$$= \sqrt{3}\int_0^1 \left(1 + (1-t)^2 e^{1-t} + t^4 e^{2t} + (1-t)^6 e^{3(1-t)}\right)dt$$

e) $C := \{\vec{r}(t) = (t, \frac{4}{t}, 5),\quad 1 \le t \le 4\}$

$$\int_C f(x,y,z)\,dx + g(x,y,z)\,dy$$
$$= \int_1^4 \left(1 + t^2 e^t + \left(\tfrac{4}{t}\right)^4 e^{2\cdot\frac{4}{t}} + 5^6 e^{3\cdot 5}\right)\cdot 1\,dt + \int_1^4 20\cdot\left(-\tfrac{4}{t^2}\right)dt$$

$$\int_C f(x,y,z)\,ds = \int_1^4 \left(1 + t^2 e^t + \left(\tfrac{4}{t}\right)^4 e^{2\cdot\frac{4}{t}} + 5^6 e^{3\cdot 5}\right)\cdot \left\|\left(1, -\tfrac{4}{t^2}, 0\right)\right\|\,dt$$
$$= \int_1^4 \left(1 + t^2 e^t + \left(\tfrac{4}{t}\right)^4 e^{2\cdot\frac{4}{t}} + 5^6 e^{3\cdot 5}\right)\cdot\sqrt{1 + \tfrac{16}{t^4}}\,dt$$

f)



Start w/ spherical coords:

$\vec{r}(u,v) = (2\cos u\sin v,\ 2\sin u\sin v,\ 2\cos v)$

$0 \le u \le 2\pi$ or $-\pi \le u \le \pi$

$? \le v \le \pi$

Need to find v here!!!

⑥

At the pt. where we want to find v we have $x^2+y^2=1$ AND $x^2+y^2+z^2=4$ AND $z \le 0$. So $1+z^2=4 \Rightarrow z^2=3 \Rightarrow z=-\sqrt{3}$

The z-coordinate of $\vec{r}(u,v) = 2\cos v$,
so $\cos v = -\frac{\sqrt{3}}{2} \Rightarrow v = \frac{5\pi}{6}$

(I figured this out using the fact that $\cos\theta$ is a decreasing fct. between 0 & π. So, the special $\measuredangle$'s in order are:
$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi$
The corresponding values are:
$1, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}, 1$)

So to summarize, we have
$\vec{r}(u,v) = (2\cos u \sin v, 2\sin u \sin v, 2\cos v)$
with $0 \le u \le 2\pi$, $\frac{5\pi}{6} \le v \le \pi$

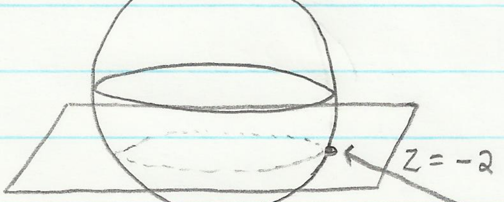
Next: $\vec{r}_u = (-2\sin u \sin v, 2\cos u \sin v, 0)$
$\vec{r}_v = (2\cos u \cos v, 2\sin u \cos v, -2\sin v)$

$\vec{r}_u \times \vec{r}_v = (-4\cos u \sin^2 v, -4\sin u \sin^2 v, -4\sin v \cos v)$

Now for v close to, but less than π, $\sin v > 0$ & $\cos v < 0$ so $-4\sin v\cos v > 0$, & so we have the right orientation. $\therefore$

$$\iint_E \vec{F}\cdot\vec{n}\,dS = \int_{v=\frac{5\pi}{6}}^{\pi}\int_{u=0}^{2\pi} (2\cos u\sin v, 4\cos u\sin v + 4\sin u\sin v\cos v, 4\sin^2 u\sin^2 v - 4\cos^2 v)\cdot(-4\cos u\sin^2 v, -4\sin u\sin^2 v, -4\sin v\cos v)\,du\,dv$$

$$\iint_E \vec{G}\cdot\vec{n}\,dS = \int_{v=\frac{5\pi}{6}}^{\pi}\int_{u=0}^{2\pi} (4\sin u\sin v\cos v,\ 4\cos u\sin v\cos v + 12\sin^2 u\sin^2 v,\ 4\sin u\cos u\sin^2 v + 4\cos v)\cdot(-4\cos u\sin^2 v,\ -4\sin u\sin^2 v,\ -4\sin v\cos v)\,du\,dv$$

g) 

As before (but w/ 4 instead of 2)
$\vec{r}(u,v) = (4\cos u\sin v, 4\sin u\sin v, 4\cos v)$
$0 \le u \le 2\pi$ or $-\pi \le u \le \pi$
are still fine.

$0 \le v \le ?$ Need to find v here.

$-2 = 4\cos v \Rightarrow \cos v = -\frac{1}{2} \Rightarrow v = \frac{2\pi}{3}$.

So $0 \le v \le \frac{2\pi}{3}$.

Now we will get after a short computation
$\vec{r}_u \times \vec{r}_v = (-16\cos u\sin^2 v, -16\sin u\sin^2 v, -16\sin v\cos v)$
This normal points inward, so we have the wrong sign.

$$\iint_E \vec{F}\cdot\vec{n}\,dS = \int_{v=0}^{\frac{2\pi}{3}}\int_{u=0}^{2\pi} (4\cos u\sin v,\ 8\cos u\sin v + 16\sin u\sin v\cos v,\ 16\sin^2 u\sin^2 v - 16\cos^2 v)\cdot(16\cos u\sin^2 v,\ 16\sin u\sin^2 v,\ 16\sin v\cos v)\,du\,dv$$

h) In this problem, since the surface in question is the boundary of a domain, life is much simpler if we use the divergence theorem:

$$\iint_E \vec{F}\cdot\vec{n}\,dS = \iiint_T \operatorname{div}\vec{F}\,dV \quad \& \quad \iint_E \vec{G}\cdot\vec{n}\,dS = \iiint_T \operatorname{div}\vec{G}\,dV$$

So $\iint_E \vec{F}\cdot\vec{n}\,dS = \iiint_T (1+z-2z)\,dV$

$$= \int_{z=0}^{6}\int_{y=0}^{\frac{30-5z}{3}}\int_{x=0}^{\frac{30-3y-5z}{2}} (1-z)\,dx\,dy\,dz$$

& $\iint_E \vec{G}\cdot\vec{n}\,dS = \int_{z=0}^{6}\int_{y=0}^{\frac{30-5z}{3}}\int_{x=0}^{\frac{30-3y-5z}{2}} (6y+2)\,dx\,dy\,dz$

Of course x, y, z can be integrated in any order.

(i) Again the divergence theorem is the way to go. Let "P" be the region that E bounds.

$\iint_E \vec{F}\cdot\vec{n}\,dS = \iiint_P (1-z)\,dV$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{5}\int_{z=0}^{25-r^2} (1-z)\,dz\,r\,dr\,d\theta$$

$$\iint_E \vec{G}\cdot\vec{n}\,dS = \int_{\theta=0}^{2\pi}\int_{r=0}^{5}\int_{z=0}^{25-r^2} (6r\sin\theta+2)\,dz\,r\,dr\,d\theta$$

(j) $\vec{r}(u,v) = (u, v, e^u(\sin v + 5))$ $\quad -1 \le u \le 5,\ -4 \le v \le 2$

$\vec{r}_u = (1, 0, e^u(\sin v + 5))$

$\vec{r}_v = (0, 1, e^u\cos v)$

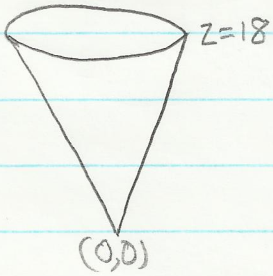
$\vec{r}_u \times \vec{r}_v = (-e^u(\sin v + 5), -e^u\cos v, 1)$

9

$$\iint_E \vec{F}\cdot\vec{n}\,dS = \int_{v=-4}^{2}\int_{u=-1}^{5} \left(u,\ 2u+ve^{u}(\sin v+5),\ v^2-e^{2u}(\sin v+5)^2\right)\cdot\left(-e^{u}(\sin v+5),\ -e^{u}\cos v,\ 1\right)du\,dv$$

$$\iint_E \vec{G}\cdot\vec{n}\,dS = \int_{v=-4}^{2}\int_{u=-1}^{5} \left(ve^{u}(\sin v+5),\ ue^{u}(\sin v+5)+3v^2,\ uv+2e^{u}(\sin v+5)\right)\cdot\left(-e^{u}(\sin v+5),\ -e^{u}\cos v,\ 1\right)du\,dv$$

(k)



$$\vec{r}(u,v) = (u\cos v,\ u\sin v,\ 2u)$$

$$\vec{r}_u = (\cos v,\ \sin v,\ 2)$$

$$\vec{r}_v = (-u\sin v,\ u\cos v,\ 0)$$

$$\vec{r}_u\times\vec{r}_v = (-2u\cos v,\ -2u\sin v,\ u)$$

$$= u(-2\cos v,\ -2\sin v,\ 1)$$

$$\|\vec{r}_u\times\vec{r}_v\| = u\sqrt{4\cos^2 v+4\sin^2 v+1} = u\sqrt{5}$$

$$\iint_E f\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{3}\left[1+u^2\cos^2 v\, e^{u\cos v}+u^4\sin^4 v\, e^{2u\sin v}+(2u)^6 e^{3(2u)}\right]\cdot u\sqrt{5}\,du\,dv$$

$$\iint_E g\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{3}\left[2u^3\cos v\sin v\right]\cdot u\sqrt{5}\,du\,dv$$

(l) start with $(5+2\cos u,\ 0,\ 2\sin u)$
after rotation around z-axis we have

$$\vec{r}(u,v) = \left((5+2\cos u)\cos v,\ (5+2\cos u)\sin v,\ 2\sin u\right)$$

$$0\le u\le 2\pi,\quad 0\le v\le 2\pi$$

$$\vec{r}_u = (-2\sin u\cos v,\ -2\sin u\sin v,\ 2\cos u)$$

$$\vec{r}_v = (-(5+2\cos u)\sin v,\ (5+2\cos u)\cos v,\ 0)$$

$$\vec{r}_u\times\vec{r}_v = \left(-10\cos u\cos v-4\cos^2 u\cos v,\ -10\cos u\sin v-4\cos^2 u\sin v,\ -2\sin u(5+2\cos u)\right)$$

(10)

$$\|\vec{r}_u \times \vec{r}_v\|^2 = \left(100\cos^2 u + 16\cos^4 u + 80\cos^3 u + 100\sin^2 u + 16\sin^2 u\cos^2 u\right)$$

$$= 100 + 80\cos^3 u + 16\cos^2 u = 4\left(25 + 20\cos^3 u + 4\cos^2 u\right)$$

$$\|\vec{r}_u \times \vec{r}_v\| = 2\sqrt{20\cos^3 u + 4\cos^2 u + 25}$$

$$\iint_E f\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{2\pi}\left(1 + (5+2\cos u)^2\cos^2 v\, e^{(5+2\cos u)\cos v} + (5+2\cos u)^4\sin^4 v\, e^{2(5+2\cos u)\sin v} + (2\sin u)^6 e^{3(2\sin u)}\right)\cdot 2\sqrt{20\cos^3 u + 4\cos^2 u + 25}\,du\,dv$$

$$\iint_E g\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{2\pi} 2(5+2\cos u)^2\cos v\sin v\sin u\cdot 2\sqrt{20\cos^3 u + 4\cos^2 u + 25}\,du\,dv$$

(m) $\vec{r}(u,v) = (u, 5\cos v, 5\sin v)$ $\quad 0 \le u \le 4,\ 0 \le v \le 2\pi$

$\vec{r}_u = (1, 0, 0)$

$\vec{r}_v = (0, -5\sin v, 5\cos v)$

$\vec{r}_u \times \vec{r}_v = (0, -5\cos v, -5\sin v)$

$\|\vec{r}_u \times \vec{r}_v\| = 5$

$$\iint_E f\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{4}\left(1 + u^2 e^u + 5^4\cos^4 v\, e^{2(5\cos v)} + 5^6\sin^6 v\, e^{3(5\sin v)}\right)\cdot 5\,du\,dv$$

$$\iint_E g\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{4}(25u\cos v\sin v)\cdot 5\,du\,dv$$

(n) $\vec{r}(u,v) = (2\cos u\sin v, 3\sin u\sin v, 5\cos v)$

$\vec{r}_u = (-2\sin u\sin v, 3\cos u\sin v, 0)$

$\vec{r}_v = (2\cos u\cos v, 3\sin u\cos v, -5\sin v)$

$\vec{r}_u \times \vec{r}_v = (-15\cos u\sin^2 v, -10\sin u\sin^2 v, -6\sin v\cos v)$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{15^2\cos^2 u\sin^4 v + 10^2\sin^2 u\sin^4 v + 6^2\sin^2 v\cos^2 v}$

⑪

$$\iint_E f\,dS = \int_{v=0}^{\pi}\int_{u=0}^{2\pi} \Big(1 + 4\cos^2 u \sin^2 v\, e^{2\cos u \sin v} + 3^4 \sin^4 u \sin^4 v\, e^{2\cdot(3\sin u \sin v)} + 5^6\cos^6 v\, e^{3(5\cos v)}\Big)\sqrt{15^2\cos^2 u \sin^4 v + 10^2 \sin^2 u \sin^4 v + 6^2 \sin^2 v \cos^2 v}\; du\,dv$$

$$\iint_E g\,dS = \int_{v=0}^{\pi}\int_{u=0}^{2\pi} 30(\sin u \cos u \sin^2 v \cos v)\cdot \sqrt{15^2\cos^2 u \sin^4 v + 10^2 \sin^2 u \sin^4 v + 6^2 \sin^2 v \cos^2 v}\; du\,dv$$