Hints and Answers to Test 3
This answer key is for a mix of versions A and B
Math 222 Fall 2018

1. Compute the mass of a solid E given by

$$0 \le x \le z \le 6, \; 2 \le y \le 4$$

and whose density function is given by $\sigma(x, y, z) = 1 + y$.

Answer: You need to compute:

$$I := \iiint_E (1 + y) \, dV \; .$$

One reasonable way to proceed:

$$I = \int_{z=0}^{6} \int_{x=0}^{z} \int_{y=2}^{4} (1+y) \, dy \, dx \, dz = \int_{z=0}^{6} \int_{x=0}^{z} \left[\left(4 + \frac{4^2}{2}\right) - \left(2 + \frac{2^2}{2}\right) \right] \, dx \, dz$$

$$= \int_{z=0}^{6} 8z \, dz = 4 \cdot 6^2 = 144.$$

2. Let R be the region given by the inequalities:

$$y \le 0, \; x^2 + y^2 + z^2 \le 9 \; .$$

Find

$$I := \iiint_R y \, dV \; .$$

Hint: This is an integral over a half-ball, so spherical coordinates are the best way to go. You have $0 \le \rho \le 3$, $\pi \le \theta \le 2\pi$ or (equally good) $-\pi \le \theta \le 0$, and $0 \le \varphi \le \pi$.

Setting things up in spherical coordinates:

$$I = \int_{\varphi=0}^{\pi} \int_{\theta=-\pi}^{0} \int_{\rho=0}^{3} (\rho \sin\theta \sin\varphi)(\rho^2 \sin\varphi) \, d\rho \, d\theta \, d\varphi \; .$$

Note that $y = \rho \sin\theta \sin\varphi$, and $\rho^2 \sin\varphi$ is the "stretch factor" that occurs with spherical coordinates. Both of these facts can be found on the cheat sheet. Since we are integrating over a region where the integrand is always negative, our answer will be negative. I leave the rest to you. (By the way, what I call the “stretch factor” is frequently called “the element of volume” or “the element of area” but I like my name better.)

3. Let R be the region given by the inequalities:

$$x \geq 0, \ y \geq 0, \ x^2 + y^2 \leq z \leq 9 \ .$$

Find

$$I := \iiint_R x \ dV \ .$$

Hint: This is an integral over a region where cylindrical coordinates make sense. We will have $0 \leq \theta \leq \pi/2$, $0 \leq r \leq 3$, and $r^2 \leq z \leq 9$.

With this set up we have:

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3} \int_{z=r^2}^{9} r \cos\theta \ dz \ r \ dr \ d\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3} r^2(9-r^2) \cos\theta \ dr \ d\theta.$$

Again, except for observing that this time we know that our answer had better be positive, I will leave the rest to you. (Don't forget to include that “r” which is the “stretch factor” for polar/cylindrical coordinates!)

4. Let T be the tetrahedron with vertices

$$P_1 = (3,0,0), \ P_2 = (0,2,0),$$

$$P_3 = (0,0,6), \ P_4 = (0,0,0) \ .$$

Consider the integral:

$$\iiint_T (y+1) \ dV \ .$$

(a) Find an equation for each face, and write down the four inequalities which determine this tetrahedron.

Answer: Hopefully you see that this tetrahedron is basically what you get by "lopping off" the corner of the first octant, and therefore there are three inequalities that come for free:

$$x \geq 0,\ \ y \geq 0,\ \ z \geq 0.$$

The hard part is finding the plane containing P_1, P_2, and P_3. That plane is given by:

$$2x + 3y + z = 6,$$

and the resulting inequality (designed to include P_4) is

$$2x + 3y + z \leq 6.$$

(b) Set up the integral as an iterated integral in each order where it could be computed without "chopping" the integral, and where the x integration is done first. (So x is the "inner" bound of integration.) **Do not bother to compute the integral.**

Answer: You could actually integrate this one in any of the six possible orders, but since I specified that x is the "inner" bound of integration there are only two ways to go:

$$\int_{y=0}^{2} \int_{z=0}^{6-3y} \int_{x=0}^{\frac{6-z-3y}{2}} (y+1)\, dx\, dz\, dy\ , \quad \text{and}$$

$$\int_{z=0}^{6} \int_{y=0}^{\frac{6-z}{3}} \int_{x=0}^{\frac{6-z-3y}{2}} (y+1)\, dx\, dy\, dz\ .$$

5. Find the curl and divergence of the following vector field:

(a) $\vec{F}(x, y, z) := \langle xy + yz,\ x^2y^3z^4,\ e^{x+2y+3z} \rangle$.

Answer:

$$\operatorname{div}(\vec{F}) = y + 3x^2y^2z^4 + 3e^{x+2y+3z} \; .$$

$$\operatorname{curl}(\vec{F}) = \langle 2e^{x+2y+3z} - 4x^2y^3z^3, y - e^{x+2y+3z}, 2xy^3z^4 - x - z \rangle \; .$$

6. Determine which of the following vector fields are conservative, and for any conservative vector fields, find a corresponding potential function.

   (a) $\vec{\Theta}(x, y, z) = \langle 1 + 4x^3,\ 2 + 3y^2,\ 3 + 2z \rangle$.

   (b) $\vec{F}(x, y) = \langle 5 - y\sin(xy^2),\ \cos(xy^2) - 2xy^4\sin(xy^2) + 3 \rangle$.

   (c) $\vec{G}(x, y) = \langle 5 - y^2\sin(xy^2),\ \cos(xy^2) - 2xy^3\sin(xy^2) + 3 \rangle$.

   (d) $\vec{H}(x, y) = \langle 5 - y^3\sin(xy^2),\ \cos(xy^2) - 2xy^2\sin(xy^2) + 3 \rangle$.

   Ack! After making the exam I realized what this question **should** have been. Here is the new lemon-scented version:

   6(a) Determine if the following vector field is conservative, and if so, find a potential function for it:

   $$\vec{\Theta}(x, y, z) = \langle 1 + 4x^3,\ 2 + 3y^2,\ 3 + 2z \rangle.$$

   6(b) Find α and β so that the following vector field is conservative, and then find the potential function for this vector field:

   $$\vec{F}(x, y) = \langle 5 - y^\alpha\sin(xy^2),\ \cos(xy^2) - 2xy^\beta\sin(xy^2) + 3 \rangle \; .$$

   Answer to (a) Because

   $$\vec{\Theta}(x, y, z) = \nabla(x + x^4 + 2y + y^3 + 3z + z^2)$$

   we can declare victory immediately.

Answer to (b) We assume that $\nabla f = \langle f_x, f_y \rangle = \vec{F}(x,y) = \langle F^1, F^2 \rangle$. The theorem we know says that the conclusions of Clairaut's Theorem are actually, **sufficient** to ensure that $\vec{F}(x,y)$ is conservative (in simply connected domains anyway). So we compute:

$$f_{xy} = F^1_y = -\alpha y^{\alpha-1} \sin(xy^2) - 2xy^{\alpha+1}\cos(xy^2) \quad \text{and}$$

$$f_{yx} = F^2_x = -y^2 \sin(xy^2) - 2y^\beta \sin(xy^2) - 2xy^{\beta+2}\cos(xy^2)\ .$$

So, because the only way that the coefficient of the sine terms can be equal, i,e,

$$-\alpha y^{\alpha-1} = -y^2 - 2y^\beta$$

is for α to be equal to 3 and for β to be equal to 2, and because that also leads to the equality of the cosine coefficients, and therefore equality of F^1_y and F^2_x, we must have $\alpha = 3$ and $\beta = 2$. Thus

$$\vec{F}(x,y) = \langle 5 - y^3 \sin(xy^2),\ \cos(xy^2) - 2xy^2 \sin(xy^2) + 3 \rangle\ .$$

Now after a bit of work, we obtain:

$$f(x,y) = y\cos(xy^2) + 5x + 3y + C\ .$$

Note that in the course of solving the "lemon-scented" version, we also solved the version above, and determined that $\vec{H}$ was conservative, and gave its potential function.

7. Use Fubini's Theorem to evaluate the integral:

$$I = \int_{y=0}^{16} \int_{x=\sqrt{y}}^{4} 3e^{x^3}\ dx\ dy\ .$$

Answer: After drawing the picture, hopefully you will see that

$$I = \int_{x=0}^{4} \int_{y=0}^{x^2} 3e^{x^3}\ dy\ dx = \int_0^4 3x^2 e^{x^3}\ dx = e^{64} - 1.$$