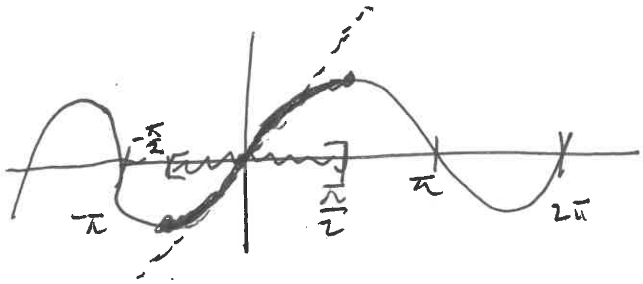
Construction of inverse trig functions

Recit #9
March 22, 2022

$\sin(x)$

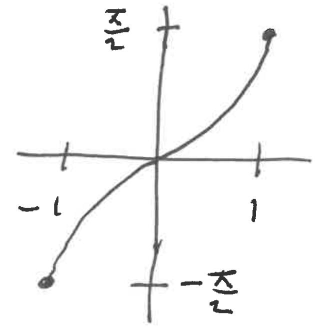
"primary branch"

$[-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$



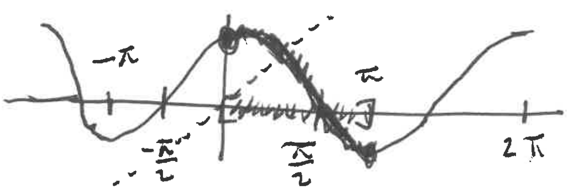
reflect primary branch $\longrightarrow$

$\sin^{-1}(x)$
or
$\arcsin(x)$
$[-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



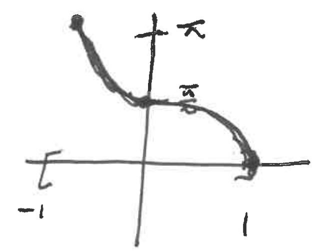
Output: Quadrants I, IV

$\cos(x)$
$[0, \pi] \longrightarrow [-1, 1]$



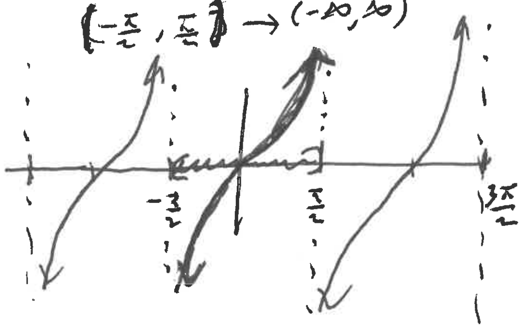
reflect primary branch $\longrightarrow$

$\cos^{-1}(x)$ or $\arccos(x)$
$[-1, 1] \longrightarrow [0, \pi]$



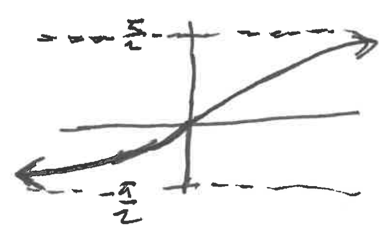
Output: Quadrants I, II

$\tan(x)$
$(-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow (-\infty, \infty)$

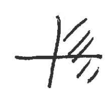


reflect primary branch $\longrightarrow$

$\tan^{-1}(x)$ or $\arctan(x)$
$(-\infty, \infty) \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

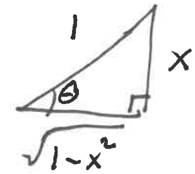


Output: Quadrants I, IV

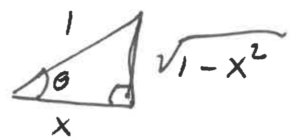


Look at my formula sheet : §2.6 to understand how Nagy's formulas came from considering triangles.

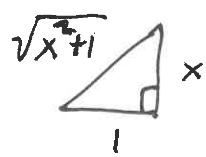
$\theta = \arcsin x \rightsquigarrow \frac{\text{opp}}{\text{hyp}} = \frac{x}{1} \rightsquigarrow$



$\theta = \arccos x \rightsquigarrow \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} \rightsquigarrow$



$\theta = \arctan x \rightsquigarrow \frac{\text{opp}}{\text{adj}} = \frac{x}{1} \rightsquigarrow$



As an exercise, check that all formulas ($6 \times 3 = 18$) hold.

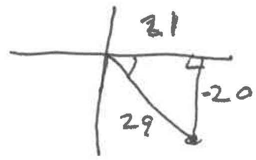
$\boxed{1-3}$

a) $\tan\left(\arcsin\left(\frac{-20}{29}\right)\right)$

arcsin output is  quadrants I, IV

Since value negative, we know its quadrant IV

Draw representative triangle



Calculate other side:

$\sqrt{29^2-(-20)^2}=21$

(20, 21, 29) are pythagorean triple

Read tan(–) off of the triangle.

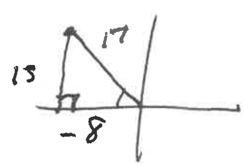
$= \boxed{\frac{-20}{21}}$

b) $\sin\left(\arccos\left(\frac{-8}{17}\right)\right)$

arc cos output 

since negative, quadrant II

Draw repr. triangle.



Calculate other side:

$\sqrt{17^2-(-8)^2}=15$

(8, 15, 17) pythag triple.

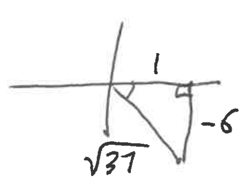
Read off sin(–)

$= \boxed{\frac{15}{17}}$

c) $\sin(\arctan(-6))$

arctan output is 

since negative, quadrant IV



$\sqrt{1^2+6^2}=\sqrt{37}$

Read off sin(–)

$= \boxed{\frac{-6}{\sqrt{37}}}$

4-6

$$\sin\left(\frac{11\pi}{4}\right) \xrightarrow{-2\pi} \sin\left(\frac{3\pi}{4}\right) = +\frac{\sqrt{2}}{2}$$

by unit circle

a) $\operatorname{asin}\left(\sin\left(\frac{11\pi}{4}\right)\right)$

$= \operatorname{asin}\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\pi}{4}$

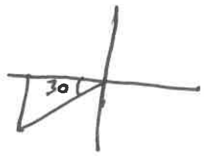
Range of arcsin (–) is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b) $\operatorname{acos}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

★ $\boxed{30°: 1, 5, 7, 11}$ ⤳ $\frac{7\pi}{6}$ is 30° in quadrant III

so

$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$



$= \operatorname{acos}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{5\pi}{6}$

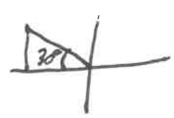
but acos (–) output is

So output will be 30° in quadrant II.

which is $\frac{5\pi}{6}$

c) $\operatorname{atan}\left(\tan\left(\frac{5\pi}{6}\right)\right)$

$\frac{5\pi}{6}$ is 30° in quadrant II

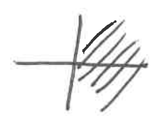


so $\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ or $-\frac{\sqrt{3}}{3}$

$= \operatorname{atan}\left(-\frac{1}{\sqrt{3}}\right)$

$= -\frac{\pi}{6}$

but atan(–) output is



So output will be 30° in quadrant IV

which is $-\frac{\pi}{6}$

! ✱ Not $\frac{11\pi}{6}$ b/c range of arctan (–) is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

7-12

For simplicity, (b) will use the same interval: $[-4\pi, 4\pi]$

"Nice" cases ~ have symmetry so 1 family of solutions.

i) $\cos x = -1$

a) $\{\pi + 2\pi k \mid k \in \mathbb{Z}\}$

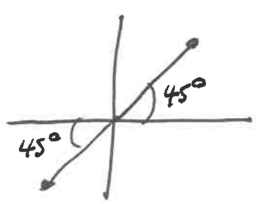
b) $\{-3\pi, -\pi, \pi, 3\pi\}$

ii) $\sin x = 0$

a) $\{0 + \pi k \mid k \in \mathbb{Z}\}$

b) $\{-3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi\}$

iii) $\tan x = 1$

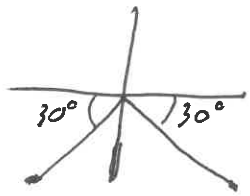


a) $\{\frac{\pi}{4} + \pi k \mid k \in \mathbb{Z}\}$

b) $\{-\frac{15\pi}{4}, -\frac{11\pi}{4}, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}\}$

Worse cases ~ multiple families of solutions

iv) $\sin x = -\frac{1}{2}$

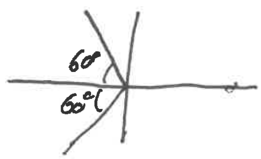


a) $\left\{ \frac{7\pi}{6} + 2\pi k \mid k \in \mathbb{Z} \right\} \cup$

$\cup \left\{ \frac{11\pi}{6} + 2\pi k \mid k \in \mathbb{Z} \right\}$

b) $\left\{ -\frac{17\pi}{6}, -\frac{13\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \right\}$

v) $\cos x = -\frac{1}{2}$



a) $\left\{ \frac{2\pi}{3} + 2\pi k \mid k \in \mathbb{Z} \right\} \cup$

$\cup \left\{ \frac{4\pi}{3} + 2\pi k \mid k \in \mathbb{Z} \right\}$

b) $\left\{ -\frac{10\pi}{3}, -\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3} \right\}$